Comparison of Potential-Based Analysis Methods for Simple and Complex Media

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Abstract—A comparison of vector and scalar potential formulations for simple and complex materials is presented. Boundary conditions for the various formulations are provided, including new scalar potential boundary conditions for layered bianisotropic gyrotropic media. Advantages and limitations of each technique are discussed in order to identify the most suitable method of analysis for a given class of material.

I. INTRODUCTION

In the solution of Maxwell’s equations, potential-based analysis methods are often sought, primarily because they provide considerable mathematical simplification. This is especially true for simple (i.e., linear, homogeneous and isotropic) media. For complex media (bianisotropic in this context), fewer potential-based methods are available. The goal here is to first provide an overview of the various potential-based methods of analysis for both simple and complex media. Boundary condition relations for material and perfect electric conducting (PEC) interfaces as well as the field recovery process are provided for each method, including new scalar-potential relations valid for bianisotropic gyrotropic materials. Advantages and limitations of each technique are discussed in order to provide a practical guide as to which method should be chosen for a given class of material. Concluding remarks are provided in the last section.

II. POTENTIAL-BASED METHODS FOR SIMPLE MEDIA

Three different potential-based methods of analysis applicable to simple media are provided in this section. The first two methods are vector potential techniques, one which employs a Lorentz gauge (the traditional technique) and another which employs a Coulomb gauge. The third method is a scalar potential-based formulation. The results for a Hertz problem of interest are not included here since it is essentially a scaled version of the Lorentz gauge approach. For the sake of brevity, only an electric current density is considered since magnetic currents can be easily accommodated via duality. A time convention of \( \exp(j\omega t) \) is assumed and suppressed.

A. Vector Potential Method - Lorentz Gauge

Maxwell’s curl and divergence equations for simple media containing electric sources only are written as

\[
\nabla \times \vec{E} = -j\omega \vec{B} = -j\omega \mu \vec{H}
\]

\[
\nabla \times \vec{H} = j\omega \vec{D} + j\omega \epsilon \vec{E} = \nabla \times \vec{A} = \mu \vec{J}
\]

Equation (2) implies that \( \vec{B} = \nabla \times \vec{A} = \mu \vec{H} \) or \( \vec{H} = \nabla \times \vec{A} / \mu \). Substitution of this result into Faraday’s law in (1) implies the electric field is given by

\[
\vec{E} = -j\omega \vec{A} - \vec{\nabla} \Phi
\]

Inserting these two relations for \( \vec{E} \) and \( \vec{H} \) into Ampere’s law in (1) and using the Lorentz gauge \( \Phi = -\nabla \cdot \vec{A} / j\omega \mu \) gives the well-known result

\[
\nabla \times \vec{A} + k^2 \vec{A} = -\mu \vec{J} \quad \text{or} \quad \nabla^2 \vec{A} + k^2 \vec{A} = -\mu \vec{J} \quad \text{...} \quad \alpha = x, y, z
\]

with \( k^2 = \omega^2 \mu \epsilon \). It is important to note that the wave equation for \( \vec{A} \) decomposes into three scalar wave equations of identical form, which leads to considerable simplification in the course of solution. The corresponding field recovery process, with aid of the Lorentz gauge, is therefore given by

\[
\vec{E} = -j\omega \vec{A} + \nabla \cdot \vec{A} / j\omega \mu, \quad \vec{H} = \nabla \times \vec{A} / \mu.
\]

This very well-known vector potential technique is almost always exclusively used for the analysis of electromagnetic problems involving simple media.

In order to take full advantage of potentials, boundary conditions in the potential domain must be determined. The three most-common boundary conditions are considered here. The first two are continuity of tangential electric and magnetic fields at a material interface. The third is tangential electric field at a PEC interface equals zero. Without loss of generality, the interface is assumed to lie in the \( x,y \)-plane. Enforcement of these material and PEC interface boundary conditions in the potential domain, using (4), leads to the respective relations [1]

\[
\vec{A}_1 = \vec{A}_2, \quad \nabla \cdot \vec{A}_1 = \nabla \cdot \vec{A}_2 \quad \text{...continuity of } \vec{E}_\text{tang}
\]

\[
\vec{A}_1 = \vec{A}_2, \quad 1 / \mu_1 \frac{\partial \vec{A}_1}{\partial z} = 1 / \mu_2 \frac{\partial \vec{A}_2}{\partial z} \quad \text{...continuity of } \vec{H}_\text{tang}
\]

\[
\vec{A}_i = 0, \quad \frac{\partial \vec{A}_i}{\partial z} = 0 \quad \text{at a PEC}
\]

where \( \vec{A}_i = \hat{x} A_x + \hat{y} A_y \).
B. Vector Potential Method - Coulomb Gauge

Based on the previous section, the formulation for the Coulomb gauge based vector potential is readily identified by setting $\Phi = 0$, leading to

$$\nabla \times \nabla \times \bar{A} - k^2 \bar{A} = \mu \bar{J}$$  \hspace{1cm} (8)

with a field recovery process dictated by the relations

$$\bar{E} = -j\omega \bar{A}, \quad \bar{H} = \frac{\nabla \times \bar{A}}{\mu}.$$  \hspace{1cm} (9)

The boundary conditions at a material and PEC interface can be identified in a similar manner as was done in the prior section, resulting in the respective potential domain relations

$$\bar{A}_1 = \bar{A}_2 \ldots \text{continuity of } \bar{E}_{\text{tang}}$$  \hspace{1cm} (10)

$$\frac{\bar{A}_1}{\mu_1} = \frac{\bar{A}_2}{\mu_2}, \quad \frac{1}{\mu_1} \frac{\partial \bar{A}_1}{\partial z} = \frac{1}{\mu_2} \frac{\partial \bar{A}_2}{\partial z} \ldots \text{continuity of } \bar{H}_{\text{tang}}$$  \hspace{1cm} (11)

$$\bar{A}_i = 0 \ldots \bar{E}_{\text{tang}} = 0 \text{ at a PEC}.$$  \hspace{1cm} (12)

This formulation is typically never used in the analysis of simple media due to the more complicated nature of the vector wave equation in (8).

C. Scalar Potential Formulation

A scalar potential formalism [2]-[3] for simple media can be identified by first decomposing Maxwell’s curl equations into transverse and longitudinal parts, leading to

$$\hat{\nabla} \times \hat{\nabla} E_z + \hat{\nabla} \times \frac{\partial \bar{E}}{\partial z} = -j\omega \mu \bar{H}_t$$  \hspace{1cm} (13)

$$\hat{\nabla} \times \hat{E}_z = -j\omega \mu H_z$$  \hspace{1cm} (14)

$$\hat{\nabla} \times \hat{E}_z + \hat{\nabla} \times \frac{\partial \bar{H}_t}{\partial z} = \bar{J}_t + j\omega \varepsilon \bar{E}_t$$  \hspace{1cm} (15)

$$\hat{\nabla} \times \hat{H}_t = \bar{J}_z + j\omega \varepsilon \bar{E}_z.$$  \hspace{1cm} (16)

The next crucial step is to expand the transverse fields and current density into lamellar and rotational parts via the 2D Helmholtz theorem, namely

$$\bar{E}_t = \hat{\nabla} \varphi + \hat{\nabla} \times \bar{z} \psi, \quad \bar{H}_t = \hat{\nabla} \pi + \hat{\nabla} \times \bar{z} \psi$$  \hspace{1cm} (17)

$$\bar{J}_t = \hat{\nabla} \mu + \hat{\nabla} \times \bar{z} v.$$  \hspace{1cm} (18)

where $\varphi, \pi, \psi, u, v$ are scalar potentials. Insertion of (17) and (18) into (13)-(16) gives the desired result

$$\begin{align*}
\Phi &= \frac{1}{j\omega \varepsilon} \frac{\partial \psi}{\partial z} \quad , \quad \pi = -\frac{1}{j\omega \mu} \frac{\partial \theta}{\partial z} \\
L_1 \psi &= s_1, \quad L_2 \theta = s_2 \\
L_1 = L_2 &= -\hat{\nabla}^2 - \omega^2 \varepsilon \mu \\
s_1 &= -\frac{\partial u}{\partial z} + J_z, \quad s_2 = -j\omega \mu v 
\end{align*}$$  \hspace{1cm} (19)

where the longitudinal fields are related to the potentials by

$$E_z = -\frac{1}{j\omega}(\hat{\nabla} \psi + J_z), \quad H_z = \frac{1}{j\omega \mu} \hat{\nabla}^2 \theta.$$  \hspace{1cm} (20)

Boundary conditions for a material and PEC interface are readily identified by taking key notice that the lamellar and rotational fields are orthogonal, leading to the relations

$$\Phi = \Phi_2, \quad \theta = \theta_2 \ldots \text{continuity of } \bar{E}_{\text{tang}}$$  \hspace{1cm} (21)

$$\pi = \pi_2, \quad \psi = \psi_2 \ldots \text{continuity of } \bar{H}_{\text{tang}}$$  \hspace{1cm} (22)

$$\Phi = 0, \quad \theta = 0 \ldots \bar{E}_{\text{tang}} = 0 \text{ at a PEC}.$$  \hspace{1cm} (23)

Summarizing, the general solutions to the wave equations in (19) for $\psi$ and $\theta$ are found first. Next, $\Phi$ and $\pi$ are computed via (19). If material and/or PEC interfaces are present, then boundary conditions must be enforced with the aid of (21)-(23). Finally, field recovery is accomplished using the relations in (17) and (20).

III. POTENTIAL-BASED METHODS FOR COMPLEX MEDIA

In this section, two potential-based methods applicable to complex (i.e., bianisotropic) media are explored. The first is a Coulomb gauge vector potential technique. The second is a scalar potential based method. Note, it appears that no Lorentz gauge based vector potential can be found that accommodates generic bianisotropic media and satisfies a simple wave equation similar to (3). Once again, for the sake of brevity, only an electric current density is considered and a time convention of $\exp(j\omega t)$ is assumed and suppressed.

A. Vector Potential Method - Coulomb Gauge

Maxwell’s curl and divergence equations for general bianisotropic media containing electric sources are written as

$$\hat{\nabla} \times \hat{E} = -j\omega \mu B = -j\omega \mu \cdot \bar{H} - j\omega \varepsilon \cdot \bar{E}$$  \hspace{1cm} (24)

$$\hat{\nabla} \times \hat{H} = \bar{J} + j\omega \varepsilon \cdot \bar{E} + j\omega \varepsilon \cdot \bar{H}$$  \hspace{1cm} (25)

It is assumed here that the material tensors are, in general, fully populated. Equation (25) implies that $\bar{B} = \hat{\nabla} \times \bar{A}$ and upon substitution into Faraday’s law produces, for a Coulomb gauge, the familiar result $\bar{E} = -j\omega \bar{A}$. Since $\bar{B} = \hat{\nabla} \times \bar{A} = \bar{B} \cdot \hat{\nabla} + \bar{z} \cdot \hat{\nabla}$, and $\bar{E} = -j\omega \bar{A}$, this implies $\bar{H} = \bar{\mu}^{-1} \cdot (\hat{\nabla} \times \bar{I} + j\omega \varepsilon \cdot \bar{A})$, where $\bar{I} = \bar{x} \hat{\nabla} + \bar{y} \hat{\nabla} + \bar{z} \hat{\nabla}$. Inserting these two relations for $\bar{E}$ and $\bar{H}$ into Ampere’s law in (24) gives the desired vector wave equation result

$$[(\hat{\nabla} \times \bar{I} + j\omega \varepsilon \cdot \bar{A}) \cdot \bar{\mu}^{-1} \cdot (\hat{\nabla} \times \bar{I} + j\omega \varepsilon \cdot \bar{A}) - \omega^2 \varepsilon ] \cdot \bar{A} = \bar{J}$$  \hspace{1cm} (26)

with field recovery given by the above relations, namely

$$\bar{E} = -j\omega \bar{A}, \quad \bar{H} = \bar{\mu}^{-1} \cdot (\hat{\nabla} \times \bar{I} + j\omega \varepsilon \cdot \bar{A}).$$  \hspace{1cm} (27)

This is the same result obtained in [4] which utilized an adjoint operator based derivation. Note, due to the trivial relationship between $\bar{E}, \bar{A}$ and the complicated relationship between $\bar{H}, \bar{A}$
in (27), boundary condition enforcement is best implemented directly with the field expressions $\vec{E}, \vec{H}$ and not with $A$.

B. Scalar Potential Formulation

A scalar potential formalism [2]-[3] for complex media can be identified by again decomposing Maxwell’s curl equations into transverse and longitudinal parts, leading to

$$\nabla \times \vec{z}E_z + \vec{z} \times \frac{\partial \vec{E}}{\partial z} = -j \omega \mu_0 \vec{H}_t - j \omega \varepsilon_0 \vec{E}_t$$  \hspace{1cm} (28)

$$\nabla \times \vec{E}_t = -\vec{z} j \omega \mu_0 \vec{H}_z - \vec{z} j \omega \varepsilon_0 \varepsilon_0 E_z$$  \hspace{1cm} (29)

$$\nabla \times \vec{z}H_z + \vec{z} \times \frac{\partial \vec{H}}{\partial z} = \vec{J}_t + j \omega \varepsilon_0 \varepsilon_0 \vec{E}_t + j \omega \mu_0 \vec{H}_t$$  \hspace{1cm} (30)

$$\nabla \times \vec{H}_t = \vec{z} \vec{J}_z + \vec{z} j \omega \mu_0 \vec{E}_z - \vec{z} j \omega \varepsilon_0 \varepsilon_0 \vec{H}_z.$$  \hspace{1cm} (31)

In order to obtain a successful scalar potential formulation, the material tensors must be restricted to being gyrotropic, that is,

$$\vec{k} = \kappa_I \vec{I} + j \kappa_\theta \vec{z} \vec{I} + \vec{z} \vec{z} \kappa_\theta $$ \hspace{1cm} (32)

where $\vec{I} = \vec{x} \times \vec{y}$. Expanding the transverse fields and current density into lamellar and rotational parts via (17)-(18) and insertion into (28)-(31) produces the desired scalar potential formulation, namely

$$\begin{bmatrix} L_1 & L_2 \\ L_3 & L_4 \end{bmatrix} \begin{bmatrix} \psi \\ \theta \end{bmatrix} = \begin{bmatrix} s_1 \\ s_2 \end{bmatrix}.$$  \hspace{1cm} (33)

The terms $L_1, L_2, L_3, L_4$ are second order differential operators and $s_1, s_2$ are source terms. The details of all these terms (as well as the relationships between $\Phi, \pi$ and $\psi, \theta$), can be found in [2], as they have been omitted here for the sake of brevity. The longitudinal fields are given by the relations

$$E_z = -\frac{\mu_0 (\nabla_2^2 \psi + J_z) + \varepsilon_0 \nabla_2^2 \theta}{j \omega \Delta_z}$$

$$H_z = \varepsilon_0 (\nabla_2^2 \psi + J_z) + \varepsilon_0 \nabla_2^2 \theta.$$  \hspace{1cm} (34)

where $\Delta_z = \varepsilon_0 \mu_0 - \varepsilon_0 \varepsilon_0 \varepsilon_0$. Boundary conditions for a material and PEC interface are readily identified by taking key notice that lamellar and rotational fields are orthogonal, once again leading to the trivial relations

$$\Phi_1 = \Phi_2, \theta_1 = \theta_2 \text{...continuity of } \vec{E}_t$$  \hspace{1cm} (35)

$$\pi_1 = \pi_2, \psi_1 = \psi_2 \text{...continuity of } \vec{H}_t$$  \hspace{1cm} (36)

$$\Phi = 0, \theta = 0 \text{...} \vec{E}_t \text{...at a PEC}.$$  \hspace{1cm} (37)

Summarizing, the general solutions to the wave equations in (33) for $\psi$ and $\theta$ are found. Next, $\Phi$ and $\pi$ are computed. If material and/or PEC interfaces are present, then boundary conditions are enforced with the aid of (35)-(37). Finally, field recovery is accomplished using the relations in (17) and (34).

IV. COMPARISON OF POTENTIAL-BASED METHODS

The goal of this section is to provide a comparison of the various potential-based formulations for both simple and complex media. The intention in doing this is to ultimately provide a practical guide for determining what method is best for a given material type and intended application.

A. Potential Method Comparison – Simple Media

The Lorentz gauge vector potential method is very well known and is almost exclusively used for problems involving simple media. The primary reason for this is that the vector wave equation reduces to three scalar wave equations having identical form. Since Fourier transforms are often employed in the solution process, in essence, this scalarization leads to a 1x1 matrix equation. This obviously leads to significant mathematical simplification. However, there are some drawbacks of this technique. First, the boundary conditions for a material interface in (5)-(6) are somewhat involved, thus great care must be taken for layered environments. Another drawback is the field recovery process in (4) is also involved, especially for the electric field. In particular, the grad div term in the electric field recovery must be handled carefully in order to obtain expressions that are valid both outside, and most importantly, inside any source region [1]. Also, as a result of the three scalar wave equations in (3), the solution is often cast into a component-by-component form, making it somewhat difficult to clearly identify certain field types, such as TE and TM modes. This mode identification can be quite useful in the design and excitation of practical electromagnetic structures. Finally, it appears that this Lorentz gauge vector potential technique cannot be used successfully for complex (i.e., bianisotropic) media. In the end, the advantage gained through the scalarization of the wave equation far exceeds any drawbacks, thereby making it the predominant choice for simple media applications.

In regards to simple media, the Coulomb gauge vector potential technique is rarely, if ever, used due to the more complicated vector wave equation in (8). In essence, a 3x3 matrix equation must be solved, which clearly complicates analysis when compared to the Lorentz gauge vector potential approach. However, its benefits are that the boundary conditions and field recovery process are easier to implement. The other main benefit is that it can be used in the treatment of bianisotropic media. When comparing the Lorentz and Coulomb gauge approaches, they are, in a sense, opposites. The Lorentz gauge approach has a simple wave equation but a more complicated boundary condition and field recovery process that is valid only for simple media. The Coulomb gauge approach has a much more complicated wave equation, but a simpler boundary condition and field recovery process that can be extended to accommodate complex media.

In the scalar potential formulation of (19), two scalar wave equations having different source terms must be solved, thus more initial work is involved when compared to the Lorentz gauge approach. However, this technique has some distinct advantages. First, boundary condition enforcement and field recovery are relatively easy to implement. Second, a careful examination of (19)-(20) reveals that the potentials $\psi, \theta$ are associated with $TM^2$ modes while $\Phi, \pi$ are associated with...
$TE^2$ modes. In addition, as a result of the 2D Helmholtz expansion, lamellar and rotational field components can be identified, leading to substantial improvement in physical insight. The remaining advantage of the scalar potential formulation is that it can be extended to complex media, however, the material must be restricted to being bianisotropic gyrotrropic in nature.

B. Potential Method Comparison – Complex Media

As previously discussed, the Coulomb gauge vector potential method can be extended to complex media, as was shown in (26)-(27). Its primary strength is that it can be used to analyze generic bianisotropic media having no restrictions on the form of the material property tensors. However, the wave equation is somewhat complicated (again, essentially a 3×3 matrix equation) and boundary conditions are best enforced on the electromagnetic fields directly. In addition, the field recovery process for the magnetic field is quite involved.

Another observation can be made regarding the Coulomb gauge approach. Since $\vec{E} = -j\omega \vec{A}$ or $\vec{A} = -\vec{E} / j\omega$, simple substitution of this result into (26)-(27) leads to the interesting conclusion that

$$[(\nabla \times \vec{I} - j\omega \vec{\zeta}) \cdot \hat{\mu}^{-1} \cdot (\nabla \times \vec{I} + j\omega \vec{\zeta}) - \alpha^2 \hat{E}] \cdot \vec{E} = -j\omega \vec{I} \quad (38)$$

$$\vec{H} = -\frac{\hat{\mu}^{-1} \cdot (\nabla \times \vec{I} + j\omega \vec{\zeta}) \cdot \hat{E}}{j\omega} \quad (39)$$

This is exactly the same result one obtains when working directly with Maxwell’s equations. Thus, one concludes that there is essentially no advantage gained when using a Coulomb gauge, as it is identical to working with Maxwell’s equations directly (other than a simple factor of $-j\omega$). The same conclusion holds about the Coulomb gauge for simple media.

If the media is restricted to being bianisotropic gyrotrropic, then the scalar potential formulation in (33) has distinct advantages. The formulation produces a 2×2 matrix equation (as opposed to 3×3 in the Coulomb gauge or direct field-based approach). This reduced dimensionality simplifies mathematical effort. In addition, boundary condition enforcement and field recovery are relatively easy. The expansion into lamellar and rotational components via (17)-(18) is also a benefit, especially in terms of physical insight.

V. Conclusion

A comparison of potential based analysis methods for both simple and complex media was performed. The benefits and limitations of each technique were discussed. New boundary conditions were developed that can accommodate layered bianisotropic gyrotrropic media. The Lorentz gauge remains to be the predominant choice in analyzing simple media, although the scalar potential formulation warrants attention due to its simple boundary condition relations, easy field recovery process and enhanced physical insight that it provides. Regarding bianisotropic media, one must work directly with Maxwell’s equations if the material tensors are not gyrotrropic, otherwise, the scalar potential formulation can be used to significant advantage. Future work includes implementation of these potential based methods for the analysis of various (i.e., propagation, radiation and scattering) environments involving bianisotropic media.

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REFERENCES


