Explicit Relations for the Piezoelectric Coupling
Factors of Rotated 6mm Plates

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Abstract - Class 6mm materials in the form of plates are attractive candidates for modern resonant sensor and actuator applications, because many possess desirable attributes besides piezoelectricity. In virtually all current applications of 6mm materials requiring use of the piezoelectric effect, the polar axis is either along, or perpendicular to, the major plate faces. When the polar axis is oblique to the thickness axis, however, the variety of excitation responses is broadened, and these may be used to fashion new devices. Explicit relations are given here for both the thickness- and lateral-field piezoelectric coupling coefficients driving simple thickness motions in 6mm plates.

INTRODUCTION

We treat the one-dimensional case of simple thickness modes of 6mm thin plates. For this symmetry the basal plane is isotropic, and only the angle between the plate normal and the direction of the polar axis is unique.

Usually, the plate is piezoelectrically driven by electrodes placed on the major surfaces, creating an electric field in the thickness direction; this is called thickness excitation (TE) [1]. When the exciting field lies in the plane of the plate, the situation is referred to as lateral excitation (LE) [2]. We assume that the lateral electric field is impressed along the rotated $X_1$ axis, and adopt the usual conventions defining both the crystal axial set, and the notation $(YX\ell t)\theta/\psi$ specifying the orientation of a plate aligned with a rotated coordinate set of axes. The plate thickness is taken along the rotated $X_2$ axis, which is the direction of plane wave progression.

The angles $\theta$, and $\psi$ specifying the two rotations determines the rotation matrix ($\alpha$).

We take $\alpha_n = \alpha_{2n}$ and $\zeta_n = \alpha_{1n}$. Then, using the full tensor forms of the elastic, piezoelectric, and dielectric constants, the following quantities are formed:

\[
\begin{align*}
\Gamma_{jk} &= c_{ijkl} \alpha_i \alpha_j \\
\Xi_k &= e_{nk\ell} \alpha_n \alpha_{\ell} \\
\varepsilon_{22'} &= \varepsilon_{iin} \alpha_i \alpha_n \\
\Gamma_{jk} &= \Gamma_{jk} + \Xi_j \Xi_k / \varepsilon_{22'}
\end{align*}
\]

The quantities appropriate to LE are now formed:

\[
\begin{align*}
\Lambda_k &= e_{nk\ell} \zeta_n \alpha_{\ell} \\
\varepsilon_{12'} &= \varepsilon_{iin} \zeta_i \alpha_n \\
\varepsilon_{11'} &= \varepsilon_{in} \zeta_i \zeta_n
\end{align*}
\]

The eigenvalue equation

\[
||\Gamma_{jk} - c \delta_{jk}|| = 0
\]

determines three piezoelectrically stiffened eigenstiffnesses, $c_m$, yielding the velocities $v_m = \sqrt{(c_m/p)}$, and corresponding eigenvectors $\gamma_{(m)}$, specifying the directions of particle displacement for each mode $m$, with respect to the crystallographic axes. By tradition, $m = a, b, c$, corresponds to the quasi-extensional, the fast- and slow-quasi-shear modes, respectively. The values of the piezoelectric coupling factors, $k(TE)_m$ and $k(LE)_m$ are obtained as follows. For the TE case:

\[
\begin{align*}
e_m^{(0)}(TE) &= \gamma_{(m)} \Xi_n \\
k_m(TE) &= |e_m^{(0)}(TE)| \sqrt{\varepsilon_{22'} \cdot c_m}
\end{align*}
\]
For the LE case:

\[ e_m^\circ (LE) = \gamma_n^{(m)} \Lambda_n \]  
(11)

\[ e_m^{oo} (LE) = e_m^\circ (LE) - \{\varepsilon_{12}'/\varepsilon_{22}'\} \bullet e_m^\circ (TE) \]  
(12)

\[ e_m^{oo} (LE) = \gamma_n^{(m)} \bullet \{\Lambda_n - (\varepsilon_{12}'/\varepsilon_{22}') \bullet \Xi_n\} \]  
(13)

\[ \varepsilon_{11}^{oo} = [\varepsilon_{11}' - (\varepsilon_{12}')^2/\varepsilon_{22}'] \]  
(14)

\[ k_m (LE) = |e_m^{oo} (LE)| / \sqrt{\varepsilon_{11}^{oo} \bullet e_m} \]  
(15)

In both TE and LE cases, the \(e_m\) are the piezoelectrically stiffened eigenvalues. This formalism is now applied to 6mm materials to obtain the effective piezocouplings driving the three modes, and their angular variation.

**WURTZITE STRUCTURE**

Symmetry class 6mm materials include poled ferroelectric ceramic alloys, as well as InN, ZnO, and AlN. Plate orientations are limited in their variety to a single rotation about any axis normal to the polar axis. A synopsis of the behavior of 6mm plates is given below. The polar axis is denoted \(P\); \(P'\) is the direction of projection of \(P\) onto the plane of the plate, and \(\hat{n}\) is the direction of the plate normal.

For \(P\) parallel to the plane normal (Z cut), all modes are pure, with the shears degenerate. TE drives only the extensional mode. LE drives only the shear mode, and displacement is along the LE field direction.

For \(P\) in the plane of the plate (Y cut), all three modes are pure, with one shear mode having displacement along \(P\), and the other having displacement in the plane, but normal to \(P\). TE drives only the shear mode having motion along \(P\). LE drives only the extensional mode, providing there is a component of \(E_{\text{lateral}}\) along \(P\).

When the unique axis \(P\) is neither parallel nor perpendicular to the plate normal, one shear mode remains pure, and inert to TE (polarized perpendicular to both \(P\) and to the plate normal). It may be either the ‘b’ or ‘c’ mode, depending on the material and angle. The other two modes are coupled, and TE drives both. All three modes, the pure shear and the two coupled modes, are driven by LE if \(E_{\text{lateral}}\) is neither normal nor parallel to \(P\). If \(E_{\text{lateral}}\) is normal to \(P\), only the pure shear mode, polarized normal to \(P\), (and therefore along \(E_{\text{lateral}}\)), is driven by LE. If \(E_{\text{lateral}}\) is parallel to \(P\), only the two coupled modes are driven by LE, (since they both have components of displacement along \(E_{\text{lateral}}\)), and the motions are normal to that of the inert pure shear mode, i.e., in the plane containing \(P\) and the plate normal.

For a rotated cut, with \(P\) oblique to the plate normal, the pure shear perpendicular to both \(P\) and \(\hat{n}\) is driven by LE (for a rotated Y cut, the required \(E\) is \(E_1\)); although \(k(LE)\) goes to zero at the Y cut. At the Y cut, (\(\theta = 0^\circ\)) the stiffness is \(c_{66}^E\); for the Z cut, it is \(c_{44}^E\). For \(\theta\) between \(0^\circ\) and \(90^\circ\), the stiffness for this pure mode transforms as \((c_{66}^E \cos^2(\theta) + c_{44}^E \sin^2(\theta))\).

The coupled shear and extension modes (in the plane of \(P'\) and \(\hat{n}\), for a rotated cut, are driven by LE (for a rotated Y cut, the required \(E\) is \(E_3\)). At the Y cut, the mode coupling ceases, and \(k(LE)\) drives the a mode only. At the Z cut, the mode coupling likewise ceases, but now \(k(LE)\) drives the degenerate shear polarized normal to \(P\) with isotropic azimuthal piezocoupling; the displacement is along the applied field. At the Y cut, the a mode stiffness is \(c_{11}^E\); for the Z cut, it is \(c_{33}\bar{e}_{\text{bar}} = c_{33}^E + e_{33}^2/e_{33}\). For \(\theta\) between \(0^\circ\) and \(90^\circ\), the stiffness for this mode transforms in a somewhat complicated manner.

The above results are detailed in the following tables. Table 1 gives the effective elastic stiffnesses for the three modes of Y-cut, rotated-Y-cut, and Z-cut 6mm materials. Tables 2, 3, and 4 give the corresponding piezocoupling factors. In Tables 1, 2, and 4, and in the formulas below, \(s\) stands for \(\sin(\theta)\), etc., and in Table 4, the additional abbreviations and relations hold:

\[ e_\circ^\circ (TE) = [\gamma_2^{(a)} \Xi_2 + \gamma_3^{(a)} \Xi_3] \]  
(16)

\[ e_\circ^\circ (LE) = [\gamma_2^{(a)} \Lambda_2 + \gamma_3^{(a)} \Lambda_3] \]  
(17)

\[ e_\circ^\circ (TE) = [\gamma_2^{(c)} \Xi_2 + \gamma_3^{(c)} \Xi_3] \]  
(18)

\[ e_\circ^\circ (LE) = [\gamma_2^{(c)} \Lambda_2 + \gamma_3^{(c)} \Lambda_3], \]  
(19)
where

\[ \Xi_2 = (e_{15} - e_{31}) \cos \theta \sin \theta \]
\[ \Xi_3 = [e_{15} \cos^2 \theta + e_{33} \sin^2 \theta] \]
\[ \Lambda_2 = [e_{15} \sin \theta \cos \theta \sin \psi] \]
\[ \Lambda_3 = (e_{15} - e_{33}) \cos \theta \sin \theta \sin \psi \]

\[ |e_{a^{oo}}(LE)| = |e_{a^{oo}}(LE) - (e_{12}'/e_{22}') \cdot e_{a^{oo}}(TE)| \]
\[ = \left| (\gamma_2^{(a)} \Lambda_2 + \gamma_3^{(a)} \Lambda_3) - (e_{12}'/e_{22}') \cdot \left( \gamma_2^{(a)} \Xi_2 + \gamma_3^{(a)} \Xi_3 \right) \right| \]

\[ |e_{c^{oo}}(LE)| = |e_{c^{oo}}(LE) - (e_{12}'/e_{22}') \cdot e_{c^{oo}}(TE)| \]
\[ = \left| (\gamma_2^{(c)} \Lambda_2 + \gamma_3^{(c)} \Lambda_3) - (e_{12}'/e_{22}') \cdot \left( \gamma_2^{(c)} \Xi_2 + \gamma_3^{(c)} \Xi_3 \right) \right| \]

For the Y cut, \( \gamma^{(a)} = [0, 1, 0] \) and \( \gamma^{(c)} = [0, 0, 1] \), and for the Z cut, \( \gamma^{(a)} = [0, 0, 1] \) and \( \gamma^{(c)} = [0, 1, 0] \), but for rotated orientations, a two-dimensional eigenvalue problem must be solved. The matrix consists of the elements \( \Gamma_{22bar}, \Gamma_{33bar} \), and \( \Gamma_{33bar} \), where

\[ \Gamma_{22bar} = \left[ c_{11} \cos^2 \theta + c_{44} \sin^2 \theta \right. \\
\left. + (e_{15} + e_{31})^2 \cos^2 \theta \sin^2 \theta/e_{22}' \right] \]
\[ \Gamma_{33bar} = \left[ c_{11} \cos^2 \theta + c_{33} \sin^2 \theta \right. \\
\left. + (e_{15} + e_{31}) \cos^2 \theta + e_{33} \sin^2 \theta/e_{22}' \right] \]
\[ \Gamma_{33bar} = \left[ \cos \theta \sin \theta \right. \\
\left. + (e_{15} + e_{31}) [e_{15} \cos^2 \theta + e_{33} \sin^2 \theta]/e_{22}' \right] \]

The eigenvalues are \( c_{a} \) and \( c_{c} \) and

\[ \gamma^{(a)} = [0, \gamma_2^{(a)}, \gamma_3^{(a)}] \]
\[ \gamma^{(c)} = [0, \gamma_2^{(c)}, \gamma_3^{(c)}], \]

are the eigenvectors.

The rotated permittivities are:

\[ e_{11}' = e_{11} + \Delta c^2 \theta s^2 \psi \]
\[ e_{12}' = - \Delta c^2 \theta s^2 \psi \]
\[ e_{22}' = e_{11} c^2 \theta + e_{33} s^2 \theta \]
\[ e_{11} = e_{11} - (e_{12}')^2/e_{22}' \]
\[ = e_{11} + \Delta c^2 \theta s^2 \psi (1 - \Delta s^2 \theta/e_{22}'), \]

where \( \Delta = (e_{33} - e_{11}) \).

For \( e_{33} \neq e_{11} \), \( e_{11} \) is a function of \( \psi \), (albeit usually a weak one), so \( k_m \) (LE) is not strictly proportional to \( |\cos(\psi)| \). If \( (e_{11} - e_{33}) = 0 \), then it is cosinusoidal in \( \psi \); as the difference widens, the departure grows. The \( e_{11} \) variation is independent of piezo considerations, and applies to all uniaxial crystals, for rotations in the basal plane.

**CONCLUSION**

The behavior of the simple thickness modes of plates of symmetry class 6mm materials is given for the general case where the polar axis is oblique to the plate normal. The direction of the exciting electric field can be either parallel or perpendicular to the plate normal. Formulas for the piezocoupling coefficients are given.

**REFERENCES**


Table 1. Effective elastic stiffnesses for thickness modes of class 6mm materials

<table>
<thead>
<tr>
<th>Mode</th>
<th>Y cut</th>
<th>Rotated Y cut</th>
<th>Z cut</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>all modes pure</td>
<td>pure mode assumed b</td>
<td>all modes pure</td>
</tr>
<tr>
<td>a</td>
<td>$c_{11}^{E}$</td>
<td>$c_{\text{bar}}$</td>
<td>$c_{33}^{E} + (e_{33})^{2}/e_{33}$</td>
</tr>
<tr>
<td>b</td>
<td>$c_{66}^{E}$</td>
<td>$c_{b}^{E} = c_{66}^{E} c_{b}^{E} + c_{44}^{E} s^{2} \theta$</td>
<td>$c_{44}^{E}$ degenerate mode</td>
</tr>
<tr>
<td>c</td>
<td>$c_{44\text{bar}} = c_{44}^{E} + (e_{15})^{2}/e_{11}$</td>
<td>$c_{c\text{bar}}$</td>
<td>$c_{44}^{E}$ degenerate mode</td>
</tr>
</tbody>
</table>

Table 2. Piezocoupling factors for thickness modes of Y-cut class 6mm materials

<table>
<thead>
<tr>
<th>Mode</th>
<th>TE</th>
<th>LE($E_1$)</th>
<th>LE($E_3$)</th>
<th>LE($\psi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>0</td>
<td>0</td>
<td>$</td>
<td>e_{31}</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>0</td>
<td>0</td>
<td>0</td>
</tr>
<tr>
<td>c</td>
<td>$</td>
<td>e_{15}</td>
<td>/\sqrt{(e_{11} \cdot c_{44\text{bar}})}$</td>
<td>0</td>
</tr>
</tbody>
</table>

Table 3. Piezocoupling factors for thickness modes of Z-cut class 6mm materials

<table>
<thead>
<tr>
<th>Mode</th>
<th>TE</th>
<th>LE($E_1$)</th>
<th>LE($E_3$)</th>
<th>LE($\psi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a, c</td>
<td>$</td>
<td>e_{33}</td>
<td>/\sqrt{(e_{33} \cdot c_{33\text{bar}})}$</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>$</td>
<td>e_{15}</td>
<td>/\sqrt{(e_{11} \cdot c_{44}^{E})}$</td>
</tr>
</tbody>
</table>

Table 4. Piezocoupling factors for thickness modes of rotated-Y-cut class 6mm materials

<table>
<thead>
<tr>
<th>Mode</th>
<th>TE</th>
<th>LE($E_1$)</th>
<th>LE($E_3'$)</th>
<th>LE($\psi$)</th>
</tr>
</thead>
<tbody>
<tr>
<td>a</td>
<td>$</td>
<td>e_{a}^{o}</td>
<td>/\sqrt{(e_{22} \cdot c_{\text{bar}})}$</td>
<td>0</td>
</tr>
<tr>
<td>b</td>
<td>0</td>
<td>$</td>
<td>e_{15} s\theta</td>
<td>/\sqrt{(e_{11} \cdot c_{b}^{E})}$</td>
</tr>
<tr>
<td>c</td>
<td>$</td>
<td>e_{c}^{o}</td>
<td>/\sqrt{(e_{22} \cdot c_{c\text{bar}})}$</td>
<td>0</td>
</tr>
</tbody>
</table>