Comparison of Monte Carlo Techniques for Obtaining System-Reliability Confidence Limits

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Key Words—Monte Carlo, s-Confidence limit, Gamma distribution, Weibull distribution, Logistic distribution.

Reader Aids—  
Purpose: Widen state of art  
Special math needed for explanations: Probability and statistics  
Special math needed for results: Statistics  
Results useful to: Reliability theorists

Summary & Conclusion—Digital computer techniques are developed using a) asymptotic distributions of maximum likelihood estimators, and b) a Monte Carlo technique, to obtain approximate system reliability s-confidence limits from component test data. 2-Parameter Weibull, gamma, and logistic distributions are used to model the component failures. The components can be arranged in any system configuration: series, parallel, bridge, etc., as long as one can write the equation for system reliability in terms of component reliability. Hypothetical networks of 3, 5, and 25 components are analyzed as examples. Univariate and bivariate asymptotic techniques are compared with a double Monte Carlo method. The bivariate asymptotic technique is shown to be fast and accurate. It can guide decisions during the research and development cycle prior to complete system testing and can be used to supplement system failure data.

INTRODUCTION

The problem of obtaining system s-confidence limits from subsystem test data has been studied extensively. Springer & Thompson [24-26] developed Bayesian limits for serially connected subsystems whose failure models follow the binomial or exponential distributions by using Mellin transforms. Lannon [14] and Lutton [18] used a technique based on the asymptotic s-normality of maximum likelihood (ML) estimators. Mann & Grubb [20] discussed approximately optimum s-confidence limits for series or parallel systems with binomial or exponentially distributed failure times. For binomial failure data they also discussed an asymptotic technique for logically complex s-coherent systems. Berkbigler & Byers [2] compared “two of the most widely used techniques for computing lower limits for the reliability of series subsystems, the exact Bayesian limits of Springer & Thompson [24] and the fiducial limits using Monte Carlo simulation [15]”. Locks [16, 17] developed Monte Carlo Bayesian programs for establishing interval estimates for reliability and the MTBF for a large and complex system of any modular configuration. O’Neil [22] examined some methods of estimating lower s-confidence limits on system reliability for serial systems based on subsystem test data. Gatilfe [7] used a log-gamma method to find system s-confidence limits. His analysis goes beyond the original log-gamma method which was designed solely for serial systems. Levy & Moore [15] designed a digital computer process to obtain system reliability s-confidence limits for a system composed of different elements whose failures followed different probability distributions. The distributions of the estimators of the parameters were assumed to be known for the cases considered. Moore [21] discussed a general Monte Carlo technique to obtain system reliability s-confidence limits when the distribution of the estimators is unknown. The double Monte Carlo method is related to the “bootstrap” method of B. Efron [6]. Another way to approximate the distribution of the estimators is to use the asymptotic distribution of the estimators. Harter & Moore [8-13] derived ML estimators and their covariance matrices for complete and censored samples from the s-normal, Weibull, gamma, logistic, lognormal, and first asymptotic distribution of extreme values. Bilikam & Moore [3, 4] derived ML estimators and their covariance matrices for multiple independent grouped censored samples from the Weibull distribution with failure times known and unknown.

This paper compares three Monte Carlo techniques to obtain system reliability s-confidence limits from component test data. Two of the techniques use the asymptotic s-normality property of ML estimators while the other uses a Monte Carlo technique to obtain the approximate distributions of the estimators. In addition, for systems with three components, the true s-confidence levels are compared for the two asymptotic methods with the desired s-confidence levels.

METHODOLOGY

The two assumptions common to the three methods are:
1) The underlying life distribution family is known for each component.
2) Components have been life tested. All unknown parameters of the life distributions have been estimated from the data. For the gamma and Weibull distributions it is assumed that the location parameter is known.

Notation

- DOU Double Monte Carlo Method
- BIV Bivariate Asymptotic Method
- UNI Univariate Asymptotic Method
- \( t_m \) given mission time for system
- \( n \) sample size for component test data

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maximum likelihood (ML) estimates of the parameters \((\alpha, \beta)\) obtained from component test data.

\[ \hat{R} \]

ML estimate of reliability of component

\[ V \]

asymptotic covariance matrix of bivariate distribution of \((\hat{\alpha}, \hat{\beta})\)

\[ V_R, \text{Var}(\hat{\alpha}), \text{Var}(\hat{\beta}) \]

asymptotic variance of \(\hat{R}, \hat{\alpha}, \) and \(\hat{\beta}\)

\[ Z_i \]

simulated realizations of \(s\)-independent standard normal deviates

\((\alpha, \beta)\)

simulated realizations of parameter vector \((\alpha, \beta)\)

\((\theta, \beta)\)

scale and shape parameters of Weibull distribution

\((\gamma, \alpha)\)

scale and shape parameters of gamma distribution

\((\mu, \alpha)\)

location and scale parameters of logistic distribution

Other, standard notation is given in “Information for Readers & Authors” at the rear of each issue.

**Double Monte Carlo Method** [21]. Samples are generated whose failures have the same distribution and parameters as the estimated ones and the same number of observations as the original test data. The parameters are estimated from the simulated sample by the same method as used on the original sample. The simulated values of the parameters and the mission time are substituted in the life distribution to obtain a simulated reliability for each component.

**Bivariate Asymptotic Method.** The simulated values of the parameters of the failure model are generated by (1);

\[
\begin{align*}
(\hat{\alpha}_i, \hat{\beta}_i) &= V^{1/2}(Z_{i\alpha}, Z_{i\beta}) + (\hat{\alpha}, \hat{\beta}) \\
\end{align*}
\]  (1)

The simulated values of the parameters and the mission time are substituted in the life distribution to obtain a simulated reliability for each component.

**Univariate Asymptotic Method.** The variance of a 2-parameter function of ML estimators is asymptotically—

\[
\text{Var}\{R(\hat{\alpha}, \hat{\beta})\} = (\partial R/\partial \hat{\alpha})^2 \text{Var}\{\hat{\alpha}\} + (\partial R/\partial \hat{\beta})^2 \text{Var}\{\hat{\beta}\} + 2(\partial R/\partial \hat{\alpha})(\partial R/\partial \hat{\beta}) \text{Cov}\{\hat{\alpha}, \hat{\beta}\}. 
\]  (2)

For a simulated value of the reliability we obtain an equation similar to (1)

\[ R_i = (V_R)^{1/2}Z_i + \hat{R}. \]  (3)

By inserting the system reliability equation in terms of component reliabilities and using the univariate asymptotic distribution of each \(R_i\) as the actual distribution, one could use the Monte Carlo method called “sample” [27] to obtain the distribution of system reliability and hence system \(s\)-confidence intervals corresponding to those obtained by the univariate asymptotic method.

**Farmous Systems**

Hypothetical networks of 3, 5, and 25 components are analyzed as examples with an assumed mission time of 100 hours. Five network configurations were considered as follows: 3 components in series; 1 in series with 2 in parallel; 3 components in parallel; 5-component network (see Fig. 1); and 25-component network (see Fig. 2). The life distributions are—

![Fig. 1. Systems 1, 2, 3, and 4.](image1)

![Fig. 2. 25 Component Network.](image2)
Weibull:  \( R(t_m) = \exp[-(t_m/\theta)^\alpha]; \) location parameter = 0

(4)

Logistic:  \( R(t_m) = 1 - [1 + \exp\{-\pi(t_m - \mu)/(\sqrt{3}\alpha)\}]^{-1} \)

(5)

Gamma:  \( b(u) = \frac{\exp(-u)}{u}; \) location parameter = 0.

(6)

The Edgeworth expansion in terms of s-normal Cdf is used.

The components 1-5 have life distributions:

1. Weibull:  \( \hat{\theta} = 400, \hat{B} = 2, \ n = 30 \)
2. Gamma:  \( \hat{\gamma} = 100, \hat{\alpha} = 3, \ n = 100 \)
3, 4. Logistic:  \( \hat{\mu} = 125, \hat{\delta} = 20, \ n = 100. \) The 10 smallest and 20 largest lives are censored.
5. Weibull:  \( \hat{\theta} = 210, \hat{B} = 3, \ n = 100 \)

The 3-component networks all used only components of type one. They were chosen to see the effect of increasing system reliability.

RESULTS AND MONTE CARLO SIMULATION

Monte Carlo lower s-confidence limits and central s-confidence intervals are derived as follows:

The system reliability realizations obtained for each of the above three methods are inserted in the equation for system reliability for each system to obtain a system reliability realization. The process is repeated \( N \) times. The system reliabilities are arranged in order of increasing magnitude. The ordered reliabilities partition the interval \([0, 1]\) into \( N + 1 \) equally probable intervals. We used \( N + 1 = 1000 \) for simplicity. From the ordered simulated reliabilities we can obtain any desired central s-confidence intervals or lower s-confidence limits, since the order statistics partition the range of reliability into \( N + 1 \) equally probable intervals. To increase accuracy, the three Monte Carlo simulations were performed using \( N + 1 = 1000 \) and the s-confidence limits obtained averaged. However this was not done for systems 4 and 5 for the double Monte Carlo method because of computer-time limitations.

Tables 1 and 2 compare central s-confidence intervals for 50% to 99% s-confidence levels. The lower limit of a 99% s-confidence interval is also a 99.5% 1-tailed lower s-confidence limit and similarly for the other lower s-confidence limits. Due to computational difficulties for the double Monte Carlo method the 5-component case was rerun for all three methods with components 2 and 5 following the same Weibull failure model (calling the modification: system 6) with the results presented in Table 2. From Tables 1 and 2, the results of the two asymptotic methods tend to bracket those of the double Monte Carlo method. In general the bivariate method is conservative while the univariate asymptotic method is optimistic. To shed light on the Monte Carlo variability, Table 3 compares the ML point estimate of system reliability \( R(t_m) \) with median values of the system reliability realizations.

To measure the accuracy of the asymptotic methods each method is repeated many times with simulated test data from a component with known reliability. A single repetition is called a run. The percentage of times the s-confidence intervals covered the true system reliability is compared with the desired s-confidence level in Table 4. For both methods all components were modeled by the Weibull distribution with sample size for simulated component test data \( n = 20 \) for the univariate method and average sample size of 20 for the three components for the bivariate method. Table 4 indicates the bivariate method is conservative and in addition more accurate than the univariate method and much less sensitive to degradation due to high system reliability.

Table 5 compares CPU time for the three methods on the CDC 6600. Lannon [14] wrote computer programs that computed from sample data the parameter estimates, variance-covariance matrix, square root of variance

<table>
<thead>
<tr>
<th>TABLE 1</th>
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<tr>
<td>Upper and Lower s-Confidence Limits for Reliability of 3-Component Systems</td>
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<tr>
<td>3 Components in Series</td>
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<tr>
<td>s-Confidence</td>
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matrix, and resulting s-confidence intervals using the CDC 6600 for the bivariate asymptotic method when the failures were modeled by a 2-parameter Weibull. The total computation time was 22.3 sec for 3 components and 29.5 sec for an 8-component system using a Monte Carlo sample size of 1000. Extrapolating from these results we obtained the values for the complete job for the 25-component system.

The bivariate method is fast and accurate if the component failures are modeled by 2-parameter distributions with both parameters unknown. This method can use the exact distributions of parameter estimates when they are known [15]. For very sparse sample data, the double Monte Carlo method can be used, and would be much faster than indicated by Table 5 in this case.

**APPENDIX**

The variances of \( \hat{R}(t_m) \) were calculated by using (2).

For component #1—

\[
\text{Var}\{\hat{R}_1(t_m, \hat{\theta})\} = D_1^2 \text{Var}\{\hat{B}\} + E_1^2 \text{Var}\{\hat{\theta}\} \\
+ 2D_1E_1 \text{Cov}\{\hat{\theta}, \hat{B}\} \\
\]

\( D_1 \equiv -(t_m/\hat{\theta})^2 \ln \left( \frac{t_m}{\hat{\theta}} \right) \hat{R}(t_m) \)

\( E_1 \equiv \left( \frac{\hat{B}}{\hat{\theta}} \right)(t_m/\hat{\theta})^2 \hat{R}(t_m) \)

Substituting the mission time, parameter estimates, and asymptotic variances and covariances from Harter & Moore [11], we obtain \( \sigma_{R_1} = .02877 \). Similarly for component #5 we obtain \( \sigma_{R_5} = .02224 \).

For the two i.i.d. logistic components—

\[
\text{Var}\{\hat{R}_2(t_m, \hat{\mu}, \hat{\sigma})\} = D_2^2 \text{Var}\{\hat{\sigma}\} + E_2^2 \text{Var}\{\hat{\sigma}\} \\
+ 2D_2E_2 \text{Cov}\{\hat{\sigma}, \hat{\mu}\} \\
\]

\( D_2 \equiv [\hat{R}(t_m)]^{-2}(1 - \hat{R}(t_m))^{-1} - 1]n/(\sqrt{3}\delta) \)

\( E_2 \equiv [\hat{R}(t_m)]^{-2}(1 - \hat{R}(t_m))^{-1} - 1]n(t_m - \hat{\mu})/(\sqrt{3}\delta^2) \).
Substituting the parameter values, mission time \( t_m \) and variances and covariances from Harter & Moore [12], we obtain \( \sigma_{\hat{\beta}} = \sigma_{\hat{\alpha}} = .01929 \).

For the component with a gamma failure distribution, we use the Edgeworth expansion of the Cdf in terms of the s-normal Cdf to obtain—

\[
\begin{align*}
\text{Var}\{ \hat{R}_2 \} &\approx \frac{1}{n} \left[ \frac{2\hat{\alpha}^2}{(\hat{\beta}^2 + \hat{\alpha}^2)^2} \right] \left[ \frac{2\hat{\alpha}^2}{(\hat{\beta}^2 + \hat{\alpha}^2)^2} \right] \\
&\quad - (2\hat{\alpha} - 1)(1 - \frac{\hat{\beta}}{2\sqrt{\hat{\alpha}}})(1 + 3\frac{\hat{\beta}}{2\sqrt{\hat{\alpha}}})
\end{align*}
\]

(9)

where \( \hat{\beta} = (t_m/\hat{\gamma} - \hat{\alpha}) \). Substituting the mission time and parameter estimates, we obtain \( \sigma_{\hat{R}_2} = .01638 \).

REFERENCES


[16] M.O. Locks, "Monte Carlo Bayesian system reliability—and MTBF—confidence assessment", Air Force Flight Dynamics Laboratory, Wright-Patterson AFB, Ohio, AFFDL-TR-75-144, AD-A057068. Available from NTIS (see [1]).


[23] R.B. Putz, "A Univariate Monte Carlo technique to approximate reliability confidence limits of systems with components characterized by the Weibull distribution", MS thesis (GOR/MA/79D-7), AFIT. Available from NTIS (see [1]).


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**Book Review**

**Software Tools**


Addison-Wesley Publishing Company; Reading, Massachusetts USA.

ISBN: 0-201-03669-X.

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This is a good book. It teaches good programming by example rather than by listing a set of cookbook rules which must be followed. The more you study it, the better it becomes. It is not a textbook, although it could be used to supplement the teacher's role in a course on programming. In order to use the book effectively, one needs some experience in programming.

Virtually all of the material is related to text processing, rather than, say, to compilers or mathematical problems. Some of this comes about because of the authors close relation to "the C programming language" (see the review, in this issue, of the book by that name) which has but few I/O facilities. The authors explain: "It might appear from this outline that we stress text manipulation too heavily. Yet computing is not all number-crunching, nor is it the 'compilers, assemblers and loaders' so hastily treated in many second courses in programming. A large part of what programmers do every day is text processing - editing program source, preparing input data, scanning output, writing documentation. ..." Some of the good points made by the authors (sometimes implicitly) are: 1) A well written program is its own documentation. And conversely, a program which requires extensive documentation probably could be rewritten to be much clearer - without the extensive documentation. 2) Good programming style, rather than blind application of particular rules or techniques, is what produces good programs. Bad programs can be written which follow all the rules. 3) Don't second-guess the computer, or yourself, on how to create efficient code. Don't outsmart yourself trying to create efficient code. First, write good understandable code. Then, let the computer tell you where it is spending most of its time - and thus where more-efficient code will help.

Any programmer who is not familiar with another of the authors' books: The Elements of Programming Style (McGraw Hill, 1974) ought to buy it and read it. It is inexpensive (approx $3), short, and easy to read. It was reviewed in vol R-23, 1974 Dec, pp 342-343.

Even though the examples come largely from text processing, the principles apply widely. The book is not expensive, and you can learn alot from it. You can apply the principles and style the day you begin reading the book. There is no need to turn the programming department upside down with a complete reorganization. * * *

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**The C Programming Language**

Brian W. Kernighan and Dennis M. Ritchie, 1978, $10.95 paperback, 228 pp.

Prentice-Hall, Inc.; Englewood Cliffs, N J 07632 USA.


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New small programming languages are being touted all over the land. Pascal is receiving a big push, perhaps because of its suitability for microcomputers which are likewise becoming ubiquitous. This is THE book on the C language. Appendix A defines the language. "C" has not been used widely enough to become endowed with all the variants which wide use creates.

Some claim that "C" is much better than Pascal because it allows the programmer more flexibility and introduces some shorthand notation for widely used, cumbersome operations (shades of APL). A favorable review of "C", by Tim Chase, appeared in Information System News, Monday 1980 May 19, p 24. Chase states that "C" can now be implemented on IBM/370, DEC PDP/11, and the HP/1000.

Undoubtedly "C" will be a useful language for creating high quality programs. Just as undoubtedly, it can be used to create very bad programs. There is a running argument (for reliable software) over how much flexi-bility a programmer should be allowed by the programming language, and how compact the language should be. APL is probably too compact. Whether the special constructs in "C" will be a bane or boon, only continued usage will tell. The C language does have many advantages; it does have some disadvantages (e.g. you have to create most of your own I/O routines, and it needs extending).

If you want to learn about "C" from the horse's mouth, this is the book to get. But beware about hoping that the programming language you use will solve all quality problems in programs. It won't.

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