and \( V(-) \) by
\[
V(-) = V_{\text{in}} \cdot \frac{(R_5 + R_B)(R_5 + R_B + R_1)}{R_6 + R_2} + \frac{V_0 \cdot F \cdot R_1}{R_5 + R_1 + R_B}
\]
where \( F \) represents the transfer function of the buffer, and \( R_B \) its output impedance.

Substitution of \( V(+) \) and \( V(-) \) in (B2) gives
\[
A_C = V_0 - AD \left( V_{\text{in}} \cdot \frac{R_6}{R_6 + R_2} + \frac{R_5 + R_B}{R_5 + R_1 + R_B} \right) - \frac{R_1}{R_5 + R_1 + R_B} \cdot F \cdot V_0
\]

or
\[
A_C = V_{\text{in}} \cdot \frac{R_6}{R_6 + R_2} + \frac{R_5 + R_B}{R_5 + R_1 + R_B} \left( \frac{V_0}{V_{\text{in}}} \cdot F \cdot \frac{R_1}{R_5 + R_1 + R_B} \right)
\]

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**REFERENCES**


**Linear Least-Squares Determination of Doppler Time Derivative for NAVSPASUR-Like Signals**

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*Abstract—* A method is derived for optimum estimation of doppler, doppler time derivative, and other parameters for doppler-type radar returns, using linear least-squares estimation procedures. It is used on radar returns from the Naval Space Surveillance System to obtain improvement of at least one order of magnitude in doppler measurement from previous practice; doppler derivative has been measured for the first time. The accurate measurement of doppler derivative (typically \( \pm 0.2 \text{ Hz}^2 \)) has enabled a resolution of the inherent geometric degeneracy in the coplanar NAVSPASUR system, to provide a significant improvement in single-pass orbit determination accuracy.

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**I. INTRODUCTION**

While the theoretical basis of maximum-likelihood/least-squares analysis is well known, it has not been universally applied to radar parameter estimation, partially because digital sampling techniques previously have been somewhat difficult to implement. In this article, the principles of linearized least-squares analysis are applied to estimate doppler and doppler time derivative of a sample of satellite echo returns received by the Naval Space Surveillance System satellite-tracking fence. This information has been used to estimate satellite velocity and acceleration for orbital determination purposes with a pre-
cision at least an order of magnitude better than previous practice. The method of linearized least-squares analysis is a method that now has wide applicability to experimental radar parameter estimation. While a moderate amount of mathematical operations upon the signal is necessary, modern digital techniques have made this practical for most applications.

II. LEAST-SQUARES THEORY

The Method of Maximum Likelihood, commonly used for the determination of experimentally derived quantities from noisy measurements, can be shown to be an unbiased, efficient estimator, and thus optimum [1]. The Method of Least-Squares [3], first derived by Gauss in the 19th century, may be shown to be the form taken by the maximum-likelihood precepts for data sets with normal error distributions, as is normally the case with noise processes. The Principle of Least-Squares postulates that the best fit to the data is obtained when the sum of the squares of the errors

\[ S = \sum_{i=1}^{n} (y_{o_i} - y_{m_i})^2 \]  

(1)

is a minimum where,

- \( i = 1 \rightarrow n \) is the set of observations,
- \( y_{o_i} \) is the observed value of the variable \( x \),
- \( y_{m_i}(p_1, \cdots, p_k, x) \) is the value of \( x \) predicted from the model, and
- \( p_1, \cdots, p_k \) are the parameters to be solved for.

If individual observations have different variances attached, the least-squares principle is generalized to minimizing the quantity

\[ S' = \sum_{i=1}^{n} \frac{(y_{o_i} - y_{m_i})^2}{\sigma_i^2} \]  

(2)

where \( \sigma_i \) is the estimated error to be attached to each observation. Now, if the function \( y_m(p_1, \cdots, p_k) \) is well behaved mathematically, minimizing \( S' \) from (2) is equivalent to setting

\[ \frac{\partial}{\partial P_j} S' = 0, \quad \text{for} \quad j = 1 \rightarrow k \]  

(3)

for each \( i \), (3) becomes

\[ \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \cdot y_{m_i} \frac{\partial y_{m_i}}{\partial p_1} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \cdot y_{o_i} \frac{\partial y_{m_i}}{\partial p_1} \]  

\[ \vdots \]  

\[ \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \cdot y_{m_i} \frac{\partial y_{m_i}}{\partial p_k} = \sum_{i=1}^{n} \frac{1}{\sigma_i^2} \cdot y_{o_i} \frac{\partial y_{m_i}}{\partial p_k} \]  

(4)

where, in general \( \left( \frac{\partial}{\partial P_j} \right) (y_{m_i}) \) is a function of \( p_1, \cdots, p_k, x \) being the associated independent variable. Equation (4) represents \( N \) equations in \( N \) unknowns, and thus can in theory be solved directly. However, for computational purposes, it is of great convenience to work with a linear equation set. This can be accomplished in one of two ways. Many observational functions are linear with respect to changes in the parameters, which is the requirement to be satisfied. For example, \( y = ax^2 \) is linear in this sense. If this does not apply, the equation set can always be linearized by differentiating and solving for small changes in the parameters. This amounts to solving for the first term in a Taylor’s series expansion in the coefficients. If the mathematical model is well behaved, the result can then be obtained by iteration. The method of linear least squares, described above, is thus of general applicability and can be solved by a standard technique.

For the linearized least-squares problem, let us define

\[ \Delta x_i = (x_{m_i} - x_{o_i}). \]

Then

\[ \Delta x_i = \frac{\partial x_{m_i}}{\partial p_1} \Delta p_1 + \cdots + \frac{\partial x_{m_i}}{\partial p_k} \Delta p_k - x_{o_i} \]  

(5)

where \( \Delta p_1, \cdots, \Delta p_k \) are small changes in the parameters \( p_1, \cdots, p_k \) (but note from the previous paragraph that many observational functions are linear in this sense). Now define

\[ \frac{\partial x_{m_i}}{\partial p_j} = C_{ij} \]  

(6)

Then, our conditional equations become simply

\[ \sum_{i=1}^{n} C_{ii} (C_{ii} \Delta p_1 + \cdots + C_{ik} \Delta p_k - \Delta x_i) = 0 \]

\[ \vdots \]

\[ \sum_{i=1}^{n} C_{ik} (C_{ik} \Delta p_1 + \cdots + C_{ik} \Delta p_k - \Delta x_i) = 0. \]  

(7)

If \( \Sigma_{i=1}^{n} \) is now denoted by [ ], the equation system becomes

\[ [C_{ii}] \Delta p_1 + \cdots + [C_{ik}] \Delta p_k = [C_{i1} \Delta x_1] \]

\[ \vdots \]

\[ [C_{ik}] \Delta p_1 + \cdots + [C_{ik}] \Delta p_k = [C_{ik} \Delta x_k]. \]  

(8)

This represents a linear equation system which is solvable by a standard matrix inversion technique. In addition, all the coefficients for each data point can be generated easily by a computer algorithm given any set of functions \( C_{ij}, \cdots, C_{ik} \).

This matrix must of course be reasonably nonsingular in order for the solution to be valid. However, this condition corresponds closely to the physical solvability of the problem.

III. APPLICATION TO DETERMINATION OF DOPPLER AND DOPPLER DERIVATIVE

For any doppler-type radar system, the mean frequency of the return is an important observable; also, the time derivative of frequency is often of equal importance, as it gives information on the curvature of the object path, which is a geometrically independent quantity. It is easily possible to apply the linear least-squares method to the determination of doppler and doppler derivative from the output of any real radar receiver, provided only that the doppler frequency is adequately sampled. Two basic cases are distinguishable mathematically, depending on whether the output is frequency or phase. The phase measurement system is intrinsically more accurate, if available, but may not always be possible to use.
If the intrinsic output of the measure is frequency, the representation of the signal is of the form

$$f(t) = f_o + K(t - t_o)$$

where,

- \( f_o \) is the constant frequency term,
- \( K \) is the doppler time derivative,
- \( t \) is the independent variable,
and

- \( t_o \) is a reference time, usually chosen near the middle of the signal.

Least-squares theory is then applied to solve for the parameters \( f_o \) and \( K \). The coefficients are

$$C_1 = \frac{\partial f}{\partial f_o} = 1$$

$$C_2 = \frac{\partial f}{\partial K} = (t - t_o)$$

and a two-variable least squares is used.

Phase data, if available, is simply the integral of the frequency. It is therefore preferable to use phase data, because the noise on an integrated quantity is always less than on its derivative. We have

$$\Phi(t) = \int f \, dt = f_o (t - t_o) + K \frac{(t - t_o)^2}{2} + C_o$$

where \( C_o \) represents an initial phase constant which must normally be solved from the data. A third term must thus be included in the least-squares solution, which results in the following coefficients:

$$C_1 = \frac{\partial \phi}{\partial f_o} = (t - t_o)$$

$$C_2 = \frac{\partial \phi}{\partial K} = \frac{(t - t_o)^2}{2}$$

$$C_3 = \frac{\partial \phi}{\partial C_o} = 1$$

for a solution for doppler derivative parameters from phase data.

Although the phase is clearly a preferable observable to use where possible, its use is more demanding, because the phase must be tracked from one sample to another without ambiguity by a phase-tracking routine. In order to do this, the conditions below must be met. Let \( \Delta t \) be the sampling/integration interval.

Then

1) The residual frequency \( \Delta f \) must be low enough so that \( \Delta f \cdot \Delta t \leq 2\pi \); that is, the signal must rotate appreciably less than a turn during the sampling interval.

2) The signal-to-noise ratio must be greater than one, so that the sum of signal and noise vectors does not pass through the origin.

If either condition is not satisfied, the method fails in a relatively disastrous fashion. The phase-tracking method was used for the present data sample, which consisted primarily of large-amplitude signals. Although conditions 1) and 2) are restrictive, their limitations can often be circumvented by appropriate signal processing. The residual frequency can be kept within bounds by the choice of an appropriate value for the heterodyne reference frequency, and the signal-to-noise ratio can be increased by pre-averaging.

An important advantage of the method of least-squares is the capability of applying weights to each data point [2], because this allows the data from each point to be included with optimum consideration of the error to be attached. In general, important physical considerations are involved in the choice of the weighting function, which is based on a consideration of the error distribution to be expected. The choice of weighting function, it should be emphasized, is not uniquely determined, but is based on an empirical or theoretical model of the system. The simplest weight function for a radar-type signal assigns a weight of 1 when the signal is above a threshold, and 0 when it is below that threshold. A more sophisticated algorithm takes note of the fact that the maximum error in the measurement of phase due to addition of a signal vector and a noise vector may be approximately represented as

$$\sigma(M) = \sqrt{\frac{1}{(A_{SIG} - A_{NOISE})}}.$$  (13)

The error in the estimation of the frequency centroid of a signal may be represented by a similar function. The weight function is the reciprocal of the error function, so that a weight that is directly proportional to signal strength is indicated (Fig. 1). This should result in estimation of parameters in a significantly more precise fashion than the uniform weighting above. A still more realistic estimation of the weight function is obtained with a model that does not allow the predicted error to decrease below a minimum (dashed line in Fig. 1). This model includes the more realistic situation in physical systems that the observational error decreases with increased signal due to the smaller random error, but only to a certain point. Beyond this, it is commonly dominated by secondary or systematic errors and does not further decrease.
in a series of complex power spectra (including phase information), with spectral resolution of 156.25 Hz and time spacing of 6.25 ms. A series of sequential samples from a particular spectral bin were then combined to form a representation of a satellite return.

Fig. 3 shows a typical example of a satellite return. The total period covered by the graph is 1.6384 s, composed of 256 X 6.25 ms samples. Each complex spectral point from bin #14 has been converted to an amplitude-phase representation, and the phase has been connected by a phase-connection algorithm. The obvious presence of linear and quadratic terms in the phase can be seen (Fig. 3(b)); these correspond to doppler and doppler derivative, respectively. This signal had a very high signal-to-noise ratio, making it an excellent test of the mathematical method used. A least-squares procedure was performed using the points within the labeled lines, and the resultant phase fit is shown in Fig. 3(c). The rms error of fit for this sample is about 15°, and the errors associated with the doppler and doppler derivative are ±0.05 Hz and 0.3 Hz², respectively. Several other sample returns were analyzed with similar results. The values obtained for doppler and doppler derivative errors agree satisfactorily with the value of the sample variance. The error estimate for mean frequency is at least an order of magnitude better than previously obtained for this system by crude “find-the-box” methods; in fact, doppler derivative had not previously been measured at all. No attempt was made to solve for the amplitude or width by least-squares techniques, due primarily to the fact that the deep fading observed on most echoes due to irregular reflection characteristics did not seem amenable to statistical averaging procedures. The sample variance remained at about ±15° for most strong satellites. This is appreciably poorer than predicted from statistical considerations discussed above. For the purposes of this article, this is an example of the type of secondary effect that limits accuracy for high signal-to-noise ratios for any system. We attribute this primarily to irregular satellite reflections—other possibilities for this residual error floor include ionospheric irregularities and phase measurement error. An attempt to introduce weighting proportional to signal amplitude did not give appreciably improved results, which is in agreement with the error floor described above. The variance in the variance of the doppler derivative measurements is considerably greater than in the mean frequency measurements. We attribute this primarily to the fact that the doppler derivative measurement is mathematically dependent on a smaller number of samples.

V. DISCUSSION

The linear least-squares method provides a mathematically powerful method of solving for relevant parameters from experimental data. We have demonstrated its use for a particular sample of doppler-type radar returns. The method is easily adaptable to all similar systems. With the advent of modern digital techniques, this type of processing has become practical for radar data, and can result in very significant increases in the accuracy of parameter estimation resulting from the application of optimum techniques. Although this particular
type of data is relatively long in duration (about one-quarter s), this is not an inherent limitation of the method, as it is not necessary to perform the computations in real time. The only strict hardware requirement is that a digital sampling at sufficient speed be performed on the signal to be analyzed.

Although extensive theoretical analysis has been performed on optimum parameter determination methods for certain types of radar measurements, such as detection and range measurement [4], [11], [13], little has been done to apply the method to problems of the specific NAVSPASUR type. Levanon and Weinstein [8], and Levanon [7], have attacked the somewhat similar problem of determining the angle of arrival, as well as velocity, of an object illuminated with a doppler radar. They correctly point out that use of phase information, or tracking, is an optimum way to solve this problem. The present analysis differs from theirs, however, in utilizing information from only a limited time interval, as is appropriate to NAVSPASUR-type data. In addition, we solve specifically for the doppler derivative, and include experimental examples with real data from the NAVSPASUR system.

The experimental results of this program represented a very significant advance over previous methods used with the NAVSPASUR radar system. Coherent doppler integration had not been applied to analysis of data from this system previously, and doppler was determined by determining which filter of a comb the signal fell in with a precision of ±20 Hz. Doppler derivative was not determined at all. It is of interest, however, to compare the present analysis to possible alternative ap-
approaches to determine if it is indeed optimum. It has been commonly accepted knowledge for some time that the method of maximum likelihood is an optimum method of solving for experimentally derived parameters, and that the method of least-squares is the subset of this that applies to Gaussian error distributions. Kalman filtering is an approach often used in real-time systems. It can be shown, however, to be an adaptation of the maximum-likelihood method for predictive purposes [10]. Thus while its use is appropriate for many radar-type systems where a continuous best estimator is desired, it is not appropriate for the NAVSPASUR problem, where a limited population of data (one satellite pass) is fully sampled before estimation. Our remarks about coherent use of phase, however, would also apply to the Kalman-type situation, and a Kalman-type filter approach should not give discordant results, since the same physical and mathematical principles are involved. An approach such as averaging is essentially a crude application of maximum likelihood principles that gives no information about the important quantity doppler derivative. Levanon and Weinstein discuss a least-squares method that includes provision for several derivatives. This method was not used here because the relatively short duration of a NAVSPASUR pass does not necessitate additional derivatives. In this connection it should be noted that, within the constraints of the maximum likelihood method, it is possible to include a number of differences in the mathematical formulation that amount to differences in the physical hypothesis of the problem. These include the validity of Gaussian statistics, which reflects whether the least-squares subset of the maximum-likelihood method can be used, the choice of normal equation, which amounts to a choice of physical model for the system, and the *a priori* estimate of accuracy to be expected from each observation, which can be reflected in the choice of weight.

An important justification for the present work was the possibility of improving single-pass orbit determination for the NAVSPASUR system. Determination of a satellite orbit from a single pass with NAVSPASUR requires the determination of a six-element state vector consisting of the vector position and velocity, or, measureables from which these can be derived. The coplanar geometry of the NAVSPASUR array results in a geometric degeneracy which normally does not allow the determination of all six of these quantities from one pass; the off-plane velocity is normally undeterminable. This degeneracy can be resolved by data from an out-of-plane receiving station, or, as we remark here, by deriving information about the curvature of the satellite path from measurements of doppler derivative. Although an actual determination of a one-pass orbit using our data was not performed, our improved signal strengths and doppler and doppler derivative error bounds provide important support to the hypothesis that improved observation modes and/or an out-of-plane station can greatly improve the determination of off-plane velocity. Sample state vector errors were computed for a typical satellite geometry with altitude of 1200 nautical miles, inclination of 65°, and pass longitude of 99° west. The known error covariance for the appropriate geometry was used, together with error estimates for the quantities in question. The existing, or baseline, system has a measurement accuracy of ±20 Hz in doppler shift and no doppler derivative measurement. This results in a typical off-plane velocity accuracy of ±2 Hz at sea level. If the doppler measurement accuracy is improved to ±1 Hz, the Doppler measurement accuracy is improved only slightly, to ±79 km/min, indicating that the fundamental geometric degeneracy has not been improved. The most dramatic improvement in the Doppler measurement comes by including doppler derivative measurements. For a test case with only accurate doppler derivative measurement added, the Doppler measurement accuracy is improved by 25×, to 3.8 km/min. This outstanding improvement reflects the fact that doppler derivative basically measures the curvature of the satellite path, overcoming the coplanar degeneracy. Although no specific calculations were performed for our test case, it is known that the out-of-plane velocity uncertainty is the major determinant in the satellite position uncertainty volume at a later time, and thus the out-of-plane velocity improvement translates directly into a prediction accuracy improvement. We have shown that doppler derivative processing is clearly possible, with measured accuracies between 1 and 5 Hz. This results in a Doppler measurement accuracy of between 2.5 and 20 km/min. Improving doppler and other measurement accuracy adds very little if doppler derivative processing is already implemented. In any event, doppler processing is a necessary by-product of doppler derivative processing. Optimum maximum likelihood processing can be applied to the more general NAVSPASUR data analysis problem, as suggested by Kellogg [5]; however, our present work encompasses the largest portion of the analysis gain in the process by including information on the satellite-path curvature.

References