Out-of-Band Response of Reflector Antennas

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Abstract—The response of reflector antennas to out-of-band frequencies has been analyzed using physical optics. A simple approximate expression has been obtained for the effective aperture, and this expression yields both the receiving pattern and the frequency dependence of the on-axis gain. The theory has been compared with published out-of-band measurements, and the pattern agreement is good, but the measured gain falls below the theory. This discrepancy is caused by mismatch loss in the coax-to-waveguide adapter.

Key Words—Effective aperture, focal region, paraboloid, physical optics, Poynting vector, out-of-band response, receiving pattern, reflector antenna.

Index Code—110d/e, 16a/e, 113d/e, 114d/e.

I. INTRODUCTION

THE RESPONSE of antennas to out-of-band frequencies [1]–[3] plays an important role in interference and jamming problems. Reflector antennas are of particular interest because they are used so frequently and because they have a strong response to above-band frequencies. The analysis of reflector antennas at above-band frequencies is complicated by the presence of higher order modes which can propagate in typical waveguide feeds. Frequencies well below the in-band frequency are not important because they are cut off by typical waveguide feeds. Consequently, “out-of-band” will refer only to above-band frequencies throughout this paper.

An earlier out-of-band analysis of reflector antennas [4] yielded the radiation pattern for each propagating mode and included a statistical analysis based on the statistics of the higher order mode coefficients. The problem is that the statistics of the higher order modes are highly dependent on the specific feed system and are not generally known. Such problems can be avoided by analyzing reflector antennas in the receiving mode and computing only the total received power carried by all the propagating modes. This total power is easily related to a generalized effective aperture which appears to be the most convenient receiving characteristic for out-of-band frequencies. Both the receiving pattern and the frequency response are given by the effective aperture, and the results depend only on the antenna parameters. The power which is coupled from the waveguide to the detector does, of course, depend on the details of the feed system, but that portion of the problem can be analyzed separately. In any case, the total power in the waveguide is a useful upper bound for the detected power.

The organization of this report is as follows. Section II contains an analysis of the fields and the Poynting vector in the focal region of a symmetrical paraboloid for plane-wave incidence. In Section III, the Poynting vector is integrated over the aperture of the feed horn to yield the total received power. The integration can be done numerically or analytically, and a simple analytical approximation is adequate for most cases. The theory is compared with some earlier out-of-band measurements for frequencies from 3 (in band) to 10 GHz [5], and the agreement is fairly good. Section IV summarizes the results of this study and makes recommendations for future work.

II. FIELDS IN THE FOCAL REGION OF A PARABOLOID

In this section we derive expressions for the electric and magnetic fields and the Poynting vector in the focal region of a symmetrical paraboloid. Much of the previous work on the focal region fields of paraboloids [6]–[10] was directed toward design of feed systems, and only on-axis incidence was considered. More recently, Valentino and Toulios [11] computed the electric field in the focal region of an offset paraboloid for off-axis incidence. For simplicity we consider only the symmetrical paraboloid, but we extend the previous derivation [11] to include the magnetic field and the Poynting vector for off-axis incidence.

A. Physical-Optics Integration

The perfectly conducting parabolic reflector is shown in Fig. 1. The reflector has a diameter D and a focal length f. The origin of a rectangular coordinate system (x, y, z) is located at the focus of the paraboloid. Our derivation and notation follow Valentino and Toulios [11] fairly closely, but we introduce several simplifications. Because we consider the symmetrical paraboloid, the offset angle $\theta_0$ in [11] is zero. For the symmetrical paraboloid, without loss of generality, we assume the incident plane wave is incident in the xz plane ($\phi_z = 0$ in [11]). Finally, we assume that the field point is located in the focal plane ($z_f = 0$ or $\theta_2 = \pi/2$ in [11]).

The physical-optics surface current $J$ on the reflector is...
given by

$$J = 2\hat{n} \times \mathbf{H}_i$$  \hspace{1cm} (1)

where \(\hat{n}\) is the unit normal to the reflector and \(\mathbf{H}_i\) is the incident magnetic field. For plane-wave incidence, \(\mathbf{H}_i\) is given by

$$\mathbf{H}_i = \frac{\mathbf{u}_R \times \mathbf{H}_i e^{-jk_0s}}{\sqrt{\frac{1}{4\pi}}}$$  \hspace{1cm} (2)

where \(k\) is the free-space wavenumber \((= 2\pi/\lambda)\), \(\lambda\) is the wavelength, and the \(\exp(j\omega t)\) time dependence is suppressed. In all cases, \(\mathbf{u}\) denotes a unit vector. We chose the quantity \(\Omega_s\) to be zero at the center of the reflector, and it is given by

$$\Omega_s = \mathbf{u}_p \cdot \mathbf{r}_s$$  \hspace{1cm} (3)

where \(\mathbf{r}_s\) is directed from the center of the reflector.

An infinitesimal surface-current patch of area \(dS\) at a point \(P_1\) on the reflector produces electric and magnetic fields \([8], [11]\) at the point \(P_2:\nabla\times \mathbf{E} = \frac{jk}{4\pi} \left[ \mathbf{J} - \mathbf{u}_R (\nabla \times \mathbf{u}_R) \right] \frac{e^{-jk_0s}}{r} \, dS$$  \hspace{1cm} (4)

where \(\mathbf{J}\) is the intrinsic impedance of free space. As indicated in Fig. 1, \(R\) is the distance between \(P_1\) and \(P_2\), and \(\mathbf{u}_R\) is directed from \(P_1\) to \(P_2\). Also, it is assumed that \(kR\) is much greater than unity.

In order to evaluate (4), we restrict our analysis to the region near the focus \((r_2 \ll r_1)\). Thus, we can assume that \(\mathbf{u}_R = -\mathbf{u}_r\) and \(R = r_1\) except in the phase term. The required expressions for the unit normal \(\hat{n}\) and the surface differential \(dS\) have been given by Bem [12]

$$\hat{n} = \frac{\partial r_1}{\partial \theta_1} \times \frac{\partial r_1}{\partial \phi_1}$$

and

$$dS = \frac{1}{\sqrt{\frac{1}{4\pi}}} \left| \frac{\partial r_1}{\partial \theta_1} \times \frac{\partial r_1}{\partial \phi_1} \right| \, d\theta_1 \, d\phi_1.$$  \hspace{1cm} (5)

Substituting (5) into (4) and carrying out some of the differentiations, we obtain

$$dE = -j \frac{E_i}{r_1 \lambda} \frac{r_1}{r} e^{-jk_0s} (r_1^2 \sin \theta_1 d\phi_1 d\theta_1)$$

$$\times \left\{ (u_{\theta_1} \cdot \mathbf{u}_H) + \frac{1}{r_1 \sin \theta_1 \phi_1} (u_{\phi_1} \cdot \mathbf{u}_H) \right\} u_{\theta_1}$$

$$+ \left\{ (u_{\phi_1} \cdot \mathbf{u}_H) + \frac{1}{r_1 \sin \theta_1 \phi_1} (u_{\theta_1} \cdot \mathbf{u}_H) \right\} u_{\phi_1}$$

$$+ \left\{ (u_{\phi_1} \cdot \mathbf{u}_H) + \frac{1}{r_1 \sin \theta_1 \phi_1} (u_{\theta_1} \cdot \mathbf{u}_H) \right\} u_{\phi_1}$$

and

$$dH = \frac{j E_i}{r_1 \lambda} \frac{r_1}{r} e^{-jk_0s} (r_1^2 \sin \theta_1 d\phi_1 d\theta_1)$$

$$\times \left\{ (u_{\theta_1} \cdot \mathbf{u}_H) + \frac{1}{r_1 \sin \theta_1 \phi_1} (u_{\phi_1} \cdot \mathbf{u}_H) \right\} u_{\theta_1} + \left\{ (u_{\phi_1} \cdot \mathbf{u}_H) + \frac{1}{r_1 \sin \theta_1 \phi_1} (u_{\theta_1} \cdot \mathbf{u}_H) \right\} u_{\phi_1}$$

where \(E_i = \eta \mathbf{H}_i\). The expression for \(dE\) agrees with previous results [11], [12]. Note that \(dE\) and \(dH\) are orthogonal and both \(dE\) and \(dH\) are transverse to \(\mathbf{u}_r\). This situation is consistent with the large \(kR\) assumption. The expression for \(R\) can be approximated as follows:

$$R = r_1 \left[ -2 \frac{r_2}{r_1} \cos \theta_1 \cos \theta_2 + \frac{r_2^2}{r_1^2} \right]^{1/2}$$

$$= r_1 - r_2 \cos \theta_1 \cos \theta_2 + \cos \theta_1 \cos \theta_2.$$  \hspace{1cm} (7)

In (7), we have neglected terms in \(r_2^2/r_1\), and the validity of this approximation has been discussed previously [11]. The phase term \(\Omega_s\) in (3) can be written

$$\Omega_s = (f - r_1) \cos \theta_1 + r_1 \sin \theta_1 \sin \theta_2 \cos \phi_1.$$  \hspace{1cm} (8)

Also, \(r_1\) is given by

$$r_1 = \frac{2f}{1 + \cos \theta_1}.$$  \hspace{1cm} (9)

The polarization of the incident magnetic field \(\mathbf{u}_H\) can be written

$$\mathbf{u}_H = a_x u_x + a_y u_y + a_z u_z$$  \hspace{1cm} (10)

where \(a_x^2 + a_y^2 + a_z^2 = 1\). Without loss of generality, we confine the analysis to two orthogonal polarizations. For the incident magnetic field polarized in the \(xz\) plane, we have

$$a_x = \cos \theta,$$

$$a_y = 0,$$

$$a_z = -\sin \theta.$$  \hspace{1cm} (11)

For the incident magnetic field polarized perpendicular to the \(xz\) plane, we have

$$a_x = 0,$$

$$a_y = 1,$$

$$a_z = 0.$$  \hspace{1cm} (12)

In order to evaluate the integration on \(\phi_1\) from 0 to \(2\pi\) in (6), we take advantage of the fact that the integrands are periodic in \(\phi_1\). The evaluation of the \(\phi_1\) integrations is given in the Appendix, and the resultant expressions for \(E\) and \(H\) are

$$E = -2jk E_i \int_0^\infty e^{-jk M_0(x, y, z) \sin \theta_1 / 1 + \cos \theta_1}$$

$$= \frac{J}{R} \sum_{n=0}^{\infty} \frac{j^n J_n(k M_0(x, y, z) \sin \theta_1)}{n!}$$

$$+ \frac{2 J}{R} \sum_{n=0}^{\infty} \frac{j^n J_n(k M_0(x, y, z) \sin \theta_1)}{n!}$$

$$+ \frac{2 J}{R} \sum_{n=0}^{\infty} \frac{j^n J_n(k M_0(x, y, z) \sin \theta_1)}{n!}$$

\(13\)
\[ H = \frac{jkE_0}{\eta} \left[ \sin \theta_1 \right] \frac{e^{-jkM_0}}{1 + \cos \theta_1} d\theta_1 \]
\[ + \sum_{n=0}^{\infty} j^n J_n(kM_1)(D_{xc} \cos n\psi + F_{xc} \sin n\psi)u_x \]
\[ + \sum_{n=0}^{\infty} j^n J_n(kM_1)(D_{yc} \cos n\psi + F_{yc} \sin n\psi)u_y \]  

The upper limit \( \theta_m \) of the \( \theta_1 \) integration is given by [11]
\[ \theta_m = \tan^{-1}\left( \frac{D}{4f} \right), \tag{14} \]

All other quantities in (13) are defined in the Appendix.

In general, the \( \theta_1 \) integration must be performed numerically, and a computer code has been written to evaluate (13). Note that the summations truncate at \( n = 2 \) which is in contrast to the case of the offset parabola [11], [12] where the summations run to \( n = \infty \).

**B. Large \( f/D \) Approximation**

When \( f/D \) is large, then \( \theta_m \) as given by (14) is small, and small argument approximations can be used for the trigonometric functions of \( \theta_1 \). In this case, (13) simplifies to

\[ E = E_0(a_xu_x - a_yu_y)I_{lm} \]
\[ H = -\frac{E_0}{\eta} (a_xu_x - a_yu_y)I_{lm} \tag{15} \]

where

\[ E_0 = -2jkE_0e^{-jkf} \sin^2 \left( \frac{\theta_m}{2} \right) \]

and

\[ I_{lm} = \frac{\csc^2 \left( \frac{\theta_m}{2} \right)}{2} \sum_{n=0}^{\infty} J_n(k\theta_1P) \sin \theta_1 d\theta_1 \]

Note that \( E \) and \( H \) are orthogonal, and both are transverse to \( u_x \). The results in (15) and (16) are consistent with earlier approximations [8], [11].

A fairly simple physical picture can be obtained from (15) and (16). The maximum of \( I_{lm} \) occurs for \( P = 0 \), and this occurs for the following focal-plane coordinates:

\[ x_2 = f \sin \theta_0 \]

and

\[ y_2 = 0. \tag{17} \]

The point determined by (17) is essentially the geometrical-optics point for a ray incident on the center of the reflector. For \( \theta_0 \) equal to zero, the maximum fields occur at the focus, but for \( \theta_0 \) not equal to zero, the maximum is shifted as indicated by (17). Away from the maximum, the decay is more rapid for higher frequencies (larger \( k \)) and also for larger reflectors (larger \( \theta_m \)). Also, the peak electric field \( E_0 \) is larger for higher frequencies and for larger reflectors.

**C. Poynting Vector**

The Poynting vector is of particular interest because, in Section III, we will integrate it over the aperture of the feed horn in order to determine the total received power. The real Poynting vector \( S \) is given by

\[ S = \frac{1}{2} \text{Re} \left( E \times H^* \right) \tag{18} \]

where \( \text{Re} \) denotes the real part and * denotes complex conjugate. A computer code has been written to evaluate (18) using the integral expressions for \( E \) and \( H \) in (13).

For large \( f/D \), we can substitute (15) and (16) into (18) to obtain the following expression:

\[ S = \frac{1}{2} \frac{|E_0|^2}{\eta} \left( a_x^2 + a_y^2 \right) \frac{2J_1(k\theta_mP)}{k\theta_mP} u_z. \tag{19} \]

For the incident magnetic field polarized in the \( y \) direction, \( a_y = 1 \) and \( a_x = 0 \). For the magnetic field polarized in the \( xz \) plane, \( a_x = \cos \theta_0 \) and \( a_y = 0 \). Since we are interested in small scan angles, we can replace \( \cos \theta_0 \) by unity and rewrite (19) as

\[ S = u_z S_0 \left( \frac{2J_1(k\theta_mP)}{k\theta_mP} \right)^2 \]

where

\[ S_0 = \frac{1}{2} \frac{|E_0|^2}{\eta}. \tag{20} \]

Thus the approximate \( S \) is independent of the polarization of the incident field and contains only a \( z \) component \( S_z \).

Most reflector antennas do not have large \( f/D \) ratios, but the approximation in (20) turns out to be surprisingly good, even for relatively small values of \( f/D \). Fig. 2 shows \( S_z \) for three different values of \( f/D \) for the on-axis case \( \theta_0 = 0 \). The curves for \( f/D = 1.0 \) and \( 0.4 \) were computed from the general integral expressions in (13), and the curve for \( f/D = \infty \) was computed from (20). We see that the dependence on \( f/D \) is quite weak, and the large \( f/D \) approximation in (20) is adequate for most cases. For off-axis incidence \( \theta_0 \neq 0 \), (20) is also a good approximation, and Fig. 3 shows some results for \( f/D = 1 \). The large \( f/D \) curves were computed from (20), and the \( f/D = 1 \) curves were computed from (13) for \( f = 10\lambda \) and for \( H_y \) polarization. The dependence on \( f/\lambda \) and the incident polarization is fairly weak. Note that the agreement
with the large $f/D$ approximation is better for the smaller scan angle $\theta_s = 10^\circ$, but is not too bad for $\theta_s = 20^\circ$. We are not interested in very large $\theta_s$ because the entire physical-optics method becomes questionable for $\theta_s$ too large.

\textbf{III. RECEIVED POWER}

\textbf{A. Feed-Horn Response}

Initially, we consider a feed horn with an aperture of arbitrary shape. Normally feed-horn aperture dimensions are on the order of a wavelength at the in-band frequency. Consequently, the aperture dimensions can be assumed to be electrically large at the higher out-of-band frequencies. Thus we make the simplifying Kirchhoff approximation that the fields in the aperture are equal to the incident fields.

If the incident fields are nonuniform, as in the focal region of a paraboloid, then the power passing through the feed-horn aperture $P_r$ is given by the integral of the Poynting vector over the aperture

$$P_r = - \int_A S_z \, dA. \quad (21)$$

$S_z$ is the $z$ component of the incident Poynting vector, and the geometry is shown in Fig. 4. Since we assume that no power is dissipated in the walls of the horn and waveguide, the power propagating down the waveguide is also given by $P_r$ in (21). Because the waveguide will normally be multimoded at out-of-band frequencies, $P_r$ is the total received power in all the propagating modes. The simple theory in (21) does not give the individual waveguide-mode amplitudes $b_n$, but it gives the sum of the square of the modal amplitudes

$$\sum_n |b_n|^2$$

which is equal to the total power if the modes are properly normalized.

A more detailed analysis of the feed horn and the junction between the feed horn and the waveguide would be required in order to determine the individual waveguide-mode coefficients. We have stayed away from this additional complexity because a knowledge of the individual mode coefficients is probably not very useful. Any waveguide bends, transitions, or irregularities would produce mode conversion and a change in the mode coefficients. Even in the uniform waveguide, the field distribution would change along the waveguide because the individual modes have different phase velocities. In contrast, the total power remains constant along the waveguide.

For the simple case where the incident field is a uniform plane wave, $S_z$ is simply given by

$$S_z = -S_0 \cos \theta \quad (22)$$

where $S_0$ is the incident power density, and $\theta$ is the incidence angle shown in Fig. 4. In this case, the integral in (21) is easily evaluated to yield

$$P_r = P_0 \cos \theta$$

where

$$P_0 = AS_0. \quad (23)$$

Thus we have the simple result that the receiving pattern of an electrically large receiving horn is simply $\cos \theta$, and is independent of polarization and the detailed shape of the aperture. This result only holds when we consider the total multimode power, and even then it is a high-frequency approximation which neglects edge diffraction. If we consider the receiving pattern for an individual waveguide mode, then the pattern has a lobe structure as expected for an electrically large antenna. The specific case of an open-ended parallel-plate waveguide has been analyzed in order to illustrate the
difference between the mode patterns and the total power pattern [13].

B. Circular-Aperture Integration

For a feed-horn aperture of arbitrary shape, the integral for the total received power in (21) must be evaluated numerically. This numerical evaluation can be rather time consuming, but for the special case of a circular aperture we can obtain an analytical approximation to (21).

For a circular aperture of radius \( r_m \), the integral in (21) can be written

\[
P_r = -\int_0^{2\pi} \int_0^{\rho_m} S_2 \rho_2 \, d\rho_2 \, d\phi_2
\]

where

\[\rho_2 = \sqrt{x_2^2 + y_2^2}\]

\[\phi_2 = \tan^{-1} \left( \frac{y_2}{x_2} \right)\]

and \( x_2 \) and \( y_2 \) are the focal-plane coordinates as shown in Fig. 1. For the general case where \( S_2 \) must be determined by numerical integration, the \( \rho_2 \) and \( \phi_2 \) integrations must also be done numerically, and a computer program has been written for this case. For large \( f/D \), we can substitute (20) into (24) and obtain the following:

\[
P_r = -S_0 \int_0^{2\pi} \int_0^{\rho_m} \left[ \frac{2J_1(k\theta_m \rho)}{k\theta_m \rho} \right]^2 \rho_2 \, d\rho_2 \, d\phi_2.
\]

First we evaluate (25) for axial incidence \( (\theta_z = 0) \). In this case, \( P = \rho_2 \) and the \( \phi_2 \) integration simply yields a factor of \( 2\pi \):

\[
P_r = -2\pi S_0 \int_0^{\rho_m} \left[ \frac{2J_1(k\theta_m \rho_2)}{k\theta_m \rho_2} \right]^2 \rho_2 \, d\rho_2.
\]

The \( \rho_2 \) integration can be done to yield the following result:

\[
P_r = P_0 \left[ 1 - J_0^2(k\theta_m \rho_m) - J_1^2(k\theta_m \rho_m) \right]
\]

where

\[P_0 = \frac{|E|^2}{2\eta} \pi (D/2)^2\]

\( P_0 \) is the total power incident on the reflector, and the factor in brackets varies from zero to unity as the quantity \( k\theta_m \rho_m \) increases from zero to infinity. Both the power density \( S_2 \) and the received power \( P_r \) are shown in Fig. 5. The ratio \( P_r/P_0 \) is called aperture efficiency, and typical values for in-band reception are on the order of 50–90 percent.

The result in (27) has been given previously by Minnett and Thomas [8], but they did not treat the case of off-axis incidence \( (\theta_z \neq 0) \). For the case of \( \theta_z \neq 0 \), we lose the \( \phi \) symmetry and must resort to an approximate integration method. Consider first the case where \( f/\sin \theta_z \) is less than \( \rho_m \). Then the maximum intensity (or geometrical-optics point) is located at \( x_2 = f \sin \theta_z \) and \( y_2 = 0 \) inside the circle \( \rho_2 = \rho_m \), as shown in Fig. 6. In order to do the integration analytically, we change the integration area from a circle of radius \( \rho_m \) centered at the origin to two semicircles of radii \( \rho_+ \) and \( \rho_- \) centered at the geometrical-optics point, as shown in Fig. 6. In this case, the integration can be evaluated as follows:

\[
P_r = \pi S_0 \left\{ \int_0^{\rho_+} \left[ \frac{2J_1(k\theta_m \rho)}{k\theta_m \rho} \right]^2 \rho \, d\rho + \int_0^{\rho_-} \left[ \frac{2J_1(k\theta_m \rho)}{k\theta_m \rho} \right]^2 \rho \, d\rho \right\}
\]

\[
= \frac{P_0}{2} \left[ 2 - J_0^2(k\theta_m \rho_+) - J_1^2(k\theta_m \rho_+) \right.\]

\[
\left. - J_0^2(k\theta_m \rho_-) - J_1^2(k\theta_m \rho_-) \right]
\]

where \( \rho_{\pm} = \rho_m \pm f \sin \theta_z \). Note that for \( \theta_z = 0 \), (28) reduces to (27).

For the case where \( \rho/\sin \theta_z \) is greater than \( \rho_m \), we use a similar strategy as indicated in Fig. 7. In this case, the circular area is replaced by an annular sector, and the received power
integral can be written
\[ P_r \approx -2 \phi_r S_0 \left[ \rho_+ \quad - \frac{2 J_1(k \theta_m \rho)}{k \theta_m} \right]^2 \rho \, d\rho \]

\[ = P_0 \frac{\phi_r}{\pi} \left[ J_0^2(k \theta_m \rho_+) - J_0^2(k \theta_m \rho_-) \right] \]

\[ - J_0^2(k \theta_m \rho_+) - J_0^2(k \theta_m \rho_-) \]  

(29)

where

\[ \rho_+ = f \sin \theta_+ \rho_m \]

\[ \rho_- = f \sin \theta_- \rho_m \]

and

\[ \phi_c = \sin^{-1} \left( \frac{\rho_m}{f \sin \theta_+} \right). \]

It is easy to see that when \( f \sin \theta_+ \) is equal to \( \rho_m \), \( \rho_- = 0 \), \( \phi_c = \pi/2 \), and (28) and (29) yield the same result for \( P_r \). Also, it can be seen from Fig. 7 that (29) provides an upper bound for \( P_r \) because the sector includes all of the circular area and the integrand is always positive.

C. Effective Aperture

The effective aperture \( A_e \) is defined as the received power divided by the intensity of the incident plane wave [14]:

\[ A_e = P_r / \langle |E'|^2 / 2\eta \rangle = P_r / (-S_0). \]

(30)

In our case, we use the same definition but recognize that \( P_r \) is the received power contained in all the propagating modes. In general, \( A_e \) must be evaluated numerically because \( P_r \) must be evaluated numerically as indicated by (21). However, for the circular aperture, we can use the approximate expressions for \( P_r \) given by (28) and (29). In doing so, it is convenient to normalize \( A_e \) to the physical aperture \( A_p = \pi(D/2)^2 \). Then we can write the normalized effective aperture as

\[
\frac{A_e}{A_p} = \frac{1}{2} \left[ 2 - J_0^2(k \theta_m \rho_+) - J_0^2(k \theta_m \rho_-) \right] - J_0^2(k \theta_m \rho_-) - J_0^2(k \theta_m \rho_+) \]

\[ - J_0^2(k \theta_m \rho_+) - J_0^2(k \theta_m \rho_-) \]

(31)

where \( \rho_s = |f \sin \theta_+ \pm \rho_m| \), and \( \phi_c \) is defined in (29).

It is possible to study the wide-angle behavior (\( f \sin \theta_+ \gg \theta_m \)) by using the large argument approximations for the Bessel functions [15] in (31)

\[ J_0^2(x) \approx \frac{2}{\pi x} \cos^2 x \]

and

\[ J_1^2(x) \approx \frac{2}{\pi x} \sin^2 x. \]

(32)

From (31) and (32) we obtain the following asymptotic expansion for \( A_e/A_p \):

\[
\frac{A_e}{A_p} \approx \frac{4 \rho_m^2}{\pi^2 k \theta_m (f \sin \theta_+)^3}.
\]

(33)

The \( (f \sin \theta_+)^{-3} \) dependence in (33) is the same as the wide-angle dependence for the radiation pattern of a circular aperture of constant illumination [14]. The \( \rho_m^2 \) dependence is to be expected because it is proportional to the area of the receiving horn.

The angular dependence of \( A_e/A_p \) for various values of \( k \theta_m \rho_m \) is shown in Fig. 8. The limiting case of \( k \theta_m \rho_m = \infty \) corresponds to geometrical optics where the power is focused to a single spot. When \( (f/\rho_m) \sin \theta_+ \) is less than unity, the spot is inside the receiving horn, and \( A_e/A_p \) is unity. When \( (f/\rho_m) \sin \theta_+ \) is greater than unity, the spot is outside the receiving horn, and \( A_e/A_p \) is zero. For \( k \theta_m \rho_m = 10 \), the focused spot is smeared out and the pattern is widened. For \( k \theta_m \rho_m = 3 \), the pattern is widened further. For \( (f/\rho_m) \sin \theta_+ \) much greater than unity, the patterns in Fig. 8 begin to approach the asymptotic expansion given by (33).

An approximate expression for the beamwidth \( \theta_b \) can be obtained by setting \( \theta_b \) equal to the incidence angle where the first zero in the \( J_1 \) Bessel function in (20) is located at the edge of the receiving horn. Then the Poynting vector in the receiving aperture will be small, and the integrated power and effective aperture will be small. This condition can be written

\[ f \sin (\theta_b/2) = \rho_m + \rho_1 \]

(34)

where \( J_1(k \rho_1 \theta_m) = 0 \) and \( k \rho_1 \theta_m = \alpha_1 \approx 3.832 \) represent the first zero of \( J_1 \) [14].

The above equation is easily solved for \( \theta_b \):

\[
\theta_b = 2 \sin^{-1} \left( \frac{\rho_m + \alpha_1 / k \theta_m}{f} \right) \approx 2 \frac{\rho_m + \alpha_1 / k \theta_m}{f}.
\]

(35)
When $k$ is infinite, we recover the geometrical-optics result, \( \sin \theta_b = 2\rho_m/f \), which is shown in Fig. 8. As $k$ becomes smaller, \( \theta_b \) becomes larger as shown in Fig. 8. For sufficiently small $k$, we have

\[
\sin \theta_b = \frac{2\alpha_1}{k\theta_m f} = \frac{4\alpha_1}{kD} = \frac{2.44\lambda}{D}.
\]

The above result agrees with the beamwidth between the first nulls for a circular aperture with constant illumination [14].

The results in Figs. 8 were determined from the analytical approximation in (31). The accuracy of (31) has been compared with numerical integration for numerous cases. A typical comparison for $f/D = 1$ is shown in Fig. 9. The numerical integration curve was done for the incident magnetic field polarized in the $y$ direction, while the analytical approximation is independent of the incident polarization. For the numerical-integration results, there is a slight dependence on polarization. Note that the general agreement is fairly good, but that the analytical result is somewhat larger for \((f/\rho_m) \sin \theta_b \) greater than unity. This is primarily a result of the upper-bound feature of the sector area approximation in Fig. 7. The numerical integration is time consuming because it involves triple numerical integration, over \( \theta_1 \) in (13) and over \( \rho_2 \) and \( \phi_2 \) in (24). Consequently, it is simpler and more efficient to use the analytical expression in (31), and this expression is adequate for most cases.

In Figs. 10–13, we compare the analytical approximation with previously reported experimental results [5] for frequencies from 3 GHz (the in-band frequency) to 10 GHz. The antenna tested had a diameter $D$ of 1.22 m (4 ft) and an $f/D$ ratio of 0.32. The small $f/D$ ratio provided a good test for the analytical approximation which is best for large $f/D$. The feed horn was rectangular, 7.21 $\times$ 5.00 cm ($2.84 \times 1.97$ in) and, for the theoretical comparison, was modeled by a circular aperture with radius $\rho_m = 3.39$ cm to yield the same area. The pattern data in [5] were taken over a range of $-40^\circ$ to $+40^\circ$ in the $H$ plane and over a smaller range (either $-5^\circ$ to $+5^\circ$ or $-10^\circ$ to $+10^\circ$) in the $E$ plane. Consequently, we show only the $H$-plane comparisons in Figs. 10–13. The pattern comparisons are for relative power or effective aperture. Strictly speaking, the theory should not be used at the in-band frequency, 3 GHz, in Fig. 10 because the feed horn and waveguide are not electrically large.

A difficulty in comparing the out-of-band measurements with the theory in Figs. 11–13 is that the coax-to-waveguide adapter has an unknown response for multimode out-of-band excitation (analysis is expected to be published [16]). The analysis in [16] indicates a very complicated response as a function of frequency and modal content. Because our theory yields the total power in the waveguide, and the adapter has a small response for some modes and some frequencies, we can expect the theory to provide an upper bound or envelope for the detected power. This seems to be the case in Figs. 11–13. It would be more appropriate to compare the theory with averaged measured patterns, but only individual measured patterns are available for comparison [5].

In Fig. 14 we show the measured gain for the same antenna
results point out the importance of the adapter in determining what portion of the total waveguide power is actually coupled into the system.

**IV. CONCLUSIONS**

The response of reflector antennas to out-of-band (above-band) frequencies has been studied in a two-step analysis. In Section II, the electric and magnetic fields and the Poynting vector in the focal region of a paraboloidal reflector have been determined by a physical-optics integration. The large $f/D$ approximation is found to be sufficiently accurate for most realistic antennas ($f/D > 0.3$), and numerical integration is therefore not required. The second step of the analysis is an integration of the Poynting vector over the aperture of the feed horn to obtain the received power. As shown in Section III, this integration can also be done by an analytical approximation in order to eliminate the need for numerical integration. The results yield a generalized effective aperture (the total received power divided by the incident power density) for the antenna, and a fairly simple expression gives both the frequency response and the receiving pattern.

The theoretical results have been compared with published measured results for a symmetrical paraboloid with $f/D = 0.32$ over a frequency range from 3 (in-band) to 10 GHz. The
A number of extensions to the basic physical-optics model have been discussed in [13]. The level of the distant sidelobes was examined using the geometrical theory of diffraction (GTD) and was found to be in approximate agreement with physical optics. The transient fields in the focal region were examined for pulse excitation, and the pulse stretching is in agreement with simple geometrical arguments. The effect of reflector roughness is examined, and the reduction in gain or effective aperture can be calculated if the surface tolerance is known. Extensions of the theory for a symmetrical reflector to the cases of offset or dual reflectors were accomplished by using equivalent focal length and diameter [17]. Other extensions and improvements to the model, such as a more precise treatment of the feed horn, are certainly possible, but the physical-optics model presented here is probably adequate for typical out-of-band EMC applications.

APPENDIX

Evaluation of the $\phi_1$ Integration

The first step in the evaluation of (6) is to write the phase term in the following form:

$$\Omega_s + R = M_0 - M_1 \cos (\phi_1 - \psi)$$  \hspace{1cm} (37)

where $M_0$, $M_1$, and $\psi$ are independent of $\phi_1$. From (7) and (8), we find that the three quantities are given by

$$M_0 = A_0 + r_1$$  \hspace{1cm} (38)

and

$$M_1 = \sqrt{(A_0 + B_s)^2 + B_c^2}$$

where

$$A_0 = (f - r_1) \cos \theta_s$$

$$A_c = r_1 \sin \theta_s \cos \phi_2$$

and

$$B_s = -r_2 \sin \theta_1 \sin \phi_2.$$

The next step is to express the amplitude terms in (6) in terms of a Fourier series in $\phi_1$. When this is done, it is found that the series truncates at $n = 2$, and $dE$ and $dH$ can be written

$$dE = -j \frac{E_0 f}{\lambda} e^{-j k(M_0 + M_1 \cos \psi)} \cdot \left\{ \sum_{n=0}^{2} [C_{en} \cos n\phi_1 + S_{en} \sin n\phi_1] u_x \right\}$$

$$+ \sum_{n=0}^{2} [C_{en} \cos n\phi_1 + S_{en} \sin n\phi_1] u_y$$

$$+ \sum_{n=0}^{2} [C_{en} \cos n\phi_1 + S_{en} \sin n\phi_1] u_z$$

$$\cdot \frac{2 \sin \theta_1}{1 + \cos \theta_1} d\theta_1 d\phi_1$$  \hspace{1cm} (39)

and

$$dH = -j \frac{E_0 f}{\lambda} e^{-j k(M_0 + M_1 \cos \psi)} \cdot \left\{ \sum_{n=0}^{2} [D_{en} \cos n\phi_1 + F_{en} \sin n\phi_1] u_x \right\}$$

$$+ \sum_{n=0}^{2} [D_{en} \cos n\phi_1 + F_{en} \sin n\phi_1] u_y$$

$$+ \sum_{n=0}^{2} [D_{en} \cos n\phi_1 + F_{en} \sin n\phi_1] u_z$$

$$\cdot \frac{2 \sin \theta_1}{1 + \cos \theta_1} d\theta_1 d\phi_1.$$

The Fourier coefficients are obtained from the terms inside the brackets in (6) and found to be

$$C_{e0} = a_x \left( \cos \theta_1 + 1/2 \frac{\sin^2 \theta_1}{1 + \cos \theta_1} \right), \quad S_{e0} = 0$$

$$C_{e1} = 0, \quad S_{e1} = -a_x \frac{\sin \theta_1}{1 + \cos \theta_1}$$

$$C_{e2} = -1/2a_x \frac{\sin^2 \theta_1}{1 + \cos \theta_1}, \quad S_{e2} = 1/2a_x \frac{\sin^2 \theta_1}{1 + \cos \theta_1}$$

$$C_{s0} = a_x \left( \cos \theta_1 + 1/2 \frac{\sin^2 \theta_1}{1 + \cos \theta_1} \right), \quad S_{s0} = 0$$

$$C_{s1} = a_x \frac{\sin \theta_1}{1 + \cos \theta_1}, \quad S_{s1} = 0$$

$$C_{s2} = -1/2a_x \frac{\sin^2 \theta_1}{1 + \cos \theta_1}, \quad S_{s2} = -a_x \frac{\sin^2 \theta_1}{2} \frac{1}{1 + \cos \theta_1}$$

$$C_{g0} = 0, \quad S_{g0} = 0$$

$$C_{g1} = -a_x \sin \theta_1, \quad S_{g1} = a_x \sin \theta_1$$

$$C_{g2} = 0, \quad S_{g2} = 0$$  \hspace{1cm} (40)
and
\[
\begin{align*}
D_{s0} &= \frac{a_z}{2} (1 + \cos \theta_1), \quad F_{s0} = 0 \\
D_{s1} &= -a_z \frac{\cos \theta_1 \sin \theta_1}{1 + \cos \theta_1}, \quad F_{s1} = 0 \\
D_{\alpha} &= -\frac{a_z}{2} \frac{\sin^2 \theta_1}{1 + \cos \theta_1}, \quad F_{\alpha} = -\frac{a_z}{2} \frac{\sin^2 \theta_1}{1 + \cos \theta_1} \\
D_{s0} &= \frac{a_z}{2} (1 + \cos \theta_1), \quad F_{s0} = 0 \\
D_{s1} &= 0, \quad F_{s1} = -a_z \frac{\sin \theta_1 \cos \theta_1}{1 + \cos \theta_1} \\
D_{\alpha} &= \frac{a_z}{2} \frac{\sin^2 \theta_1}{1 + \cos \theta_1}, \quad F_{\alpha} = \frac{a_z}{2} (\cos \theta_1 - 1) \\
D_{s0} &= a_z \frac{\sin^2 \theta_1}{1 + \cos \theta_1}, \quad F_{s0} = 0 \\
D_{s1} &= -a_z \sin \theta_1, \quad F_{s1} = -a_z \sin \theta_1 \\
D_{\alpha} &= 0, \quad F_{\alpha} = 0.
\end{align*}
\]

From Fourier-Bessel expansions [15], we can derive the following useful integral:
\[
\int_0^{2\pi} \left\{ \cos n\phi \sin n\phi \right\} e^{-j k M_1 \cos (\theta_1 - \psi)} d\phi = 2\pi j^n \begin{cases} \cos n\psi \\ \sin n\psi \end{cases} J_n(k M_1) \tag{41}
\]

where \(J_n\) is the \(n\)-th order Bessel function. By using (41) in (39), the \(\phi_1\) integrations can be carried out to yield the desired expressions for \(E\) and \(H\) in (13).