THE EFFECT OF GAUSSIAN APPROXIMATIONS AND BIT-TO-BIT ERROR DEPENDENCE ON PACKET THROUGHPUT CALCULATIONS IN DS/SSMA RADIO SYSTEMS

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I. Introduction

Direct-sequence spread-spectrum multiple-access (DS/SSMA) packet radio systems allow users which send data in bursts to share a common communication channel through two techniques: that of transmitting only during times that a message is actively being sent, and the ability to transmit messages simultaneously with other users without necessarily causing mutually destructive interference. To find accurate throughput performance for such a packet system, calculations for the probability of data bit error must account for the effects of error dependence from data bit to data bit within each packet transmitted. These error dependencies stem from the fact that the relative delays and (perhaps) carrier phases between the respective packets which are simultaneously transmitted on the multiple-access channel may be essentially fixed, which cause bit-to-bit error events to become positively correlated [1].

Various analytical approaches have been taken which attempt to find packet throughput either by ignoring this bit-to-bit error dependence or by using a worst-case bound. One of the most straightforward is to assume that the channel is noiseless if \( \lambda \) or fewer packets overlap, otherwise the channel is considered useless [2]. However, the theory of moment spaces can be used to show that the channel degradation due to increasing multiple-access interference is less abrupt when bit error events are positively correlated than that predicted by models which ignore this dependence [1]. Another commonly-used method which guarantees a lower bound on throughput (given that all signals are of equal power at each receiver) is to assume that all signals are chip and phase aligned at each receiver as well [3,4]. An added benefit of this approach is that a widely-used Gaussian approximation for this worst-case multiple-access interference situation gives accurate results for the probability of data bit error under these conditions, and the effect of bit-to-bit error dependence within the packets is accounted for by fixing chip delay and carrier phase values [4]. However, we will show that this worst-case assumption produces a very loose lower bound on the probability of packet success for packets which have random chip and phase values relative to each other.
In this paper, we use an accurate approximation to the probability of data bit error based upon the improved Gaussian approximation method shown in [1], but at reduced computational complexity from the approaches taken in either [1] or [5]. This method also allows us to analyze the throughput performance of a DS/SSMA packet network in which all packets have signature sequences that are completely random from bit-to-bit. We model the network as an infinite-user slotted ALOHA system where unsuccessful transmissions are caused entirely by multiple-access interference, and we give a procedure for calculating arbitrarily tight upper and lower throughput bounds. For simplification, we work primarily with the lower throughput bound and quantify the performance enhancement gained by incorporating block error correction capability into the packets. Although throughput is increased by using error control, the required redundancy shortens the number of data bits devoted to the message in a fixed-length packet; we take this into account by calculating a quantity called the “effective throughput.”

Next, we define network capacity as the maximum effective throughput, and we calculate capacity figures by using four different methods: a standard Gaussian approximation for bit error rates using both worst-case and random chip and phase values between all signals; and an improved Gaussian approximation to the probability of data bit error, with packet success probabilities calculated both with and without accounting for the bit-to-bit error dependence caused by the multiple-access interference. Finally, we compare the capacity of the infinite-user DS/SSMA slotted ALOHA packet network with a number of infinite-user narrow-band networks occupying an equivalent bandwidth.

II. Probability of Data Bit Error

The spread-spectrum system model used is identical to that shown in [1] and [5] for \( K \) users, in which the signature sequence chosen by each user is of infinite length, independent from chip to chip, and has \( N \) chips per data bit. The decision statistic of each coherent correlation receiver consists of a contribution from a signal destined for that receiver, called the desired signal, and contributions from signals intended for other receivers, called interfering signals, which produce multiple-access interference (MAI). We assume that all data bit errors are caused only by the MAI, allowing the effect of thermal noise to be ignored.

Central limit theorem arguments can be employed to show that if the relative delays and phases between the desired and interfering signals are fixed, and if a particular autocorrelation property of the desired signature sequence is also fixed, then the MAI can be accurately modeled as a Gaussian random variable when all interfering signature sequences are randomly generated [1]. By using a characteristic function approach developed in [6], however, we show that the MAI also converges in distribution to a Gaussian random variable for desired signature sequences that are completely random, provided once again that the relative delays and phases between the desired and interfering signals are fixed. This result leads to some important advantages: 1) probability of data bit error calculations may now be accomplished with a reduction in computational complexity of order \( N \) over that used in [1] and [5], since conditioning on the desired sequence autocorrelation property is no longer required;
2) the ability to incorporate the effect of bit-to-bit error dependence within packets is retained; and 3) since all signals have the same signature sequence structure, accurate packet throughput calculations may be performed without resorting to worst-case chip and phase alignment conditioning.

For large $N$ the sum of a set of zero-mean, conditionally independent (given the differential delays $s_k$ and phases $\phi_k$ for interfering users $k=2,\ldots,K$), jointly Gaussian random variables closely models the multiple-access interference at the desired receiver caused by these interfering users. This sum is itself zero-mean Gaussian and can be characterized by its variance.

After normalizing the desired receiver decision statistic so that the magnitude of the desired signal is $N$, the variance $\Psi$ of the multiple-access interference for any $K$ is a function of the differential delays and phases, and is given by

$$\Psi(s_2, \ldots, s_K, \phi_2, \ldots, \phi_K) = \sum_{k=2}^{K} N(1-2s_k+2s_k^2) \cos^2 \phi_k.$$  (1)

Now we can use the procedure developed in [1] to determine the distribution of this variance when the $s_i$ and $\phi_i$ are random. The resulting density function $f_\Psi(x)$ of the variance is the $(K-2)$-fold convolution of the density function $f_2(x)$ of the variance of a single interfering signal, given by

$$f_2(x) = \frac{1}{2\pi x N/2} \log \left| \frac{\sqrt{N-x} + \sqrt{N/2}}{\sqrt{N-x} - \sqrt{N/2}} \right|$$  (2)

for $0 < x < N$ and $x \neq N/2$.

At this point we are in a position to compare the probability of data bit error using two different Gaussian approximations which employ the $Q$ function, where

$$Q(x) = \frac{1}{\sqrt{2\pi}} \int_{x}^{\infty} e^{-u^2/2} \, du.$$  (3)

The first method is the widely-used standard Gaussian approximation, where the probability of data bit error is given by

$$\hat{P}_e^{(1)} = Q\left( \frac{3N}{\sqrt{K-1}} \right)$$  (4)

and the second method uses the previously-derived distribution of the MAI variance $\Psi$ in an improved Gaussian approximation with probability of data bit error

$$\hat{P}_e^{(2)} = E\left[ Q\left( \frac{N}{\sqrt{\Psi}} \right) \right] = \int_{0}^{\infty} Q\left( \frac{N}{\sqrt{\Psi}} \right) f_\Psi(x) \, dx.$$  (5)

The values for $\hat{P}_e^{(1)}$ and $\hat{P}_e^{(2)}$ are compared to bounds [5] on the actual probability of data bit error in Fig. 1 for various values of $N$ and $K$.

It is evident from Fig. 1 that $\hat{P}_e^{(1)}$ seems to converge to $\hat{P}_e^{(2)}$ as $K$ increases. Although the multiple-access interference for each interfering user is not Gaussian when $s_i$ and $\phi_i$ are random [5], we can apply the central limit theorem in a manner similar to that developed in [1] on the normalized MAI. The result shows that for large $K$ the normalized output statistic of the desired receiver can be accurately modeled as a Gaussian random variable with mean $\sqrt{N/(K-1)}$ and variance $1/3$, producing an average signal-to-noise ratio of $\sqrt{3N/(K-1)}$.
III. Probability of Packet Success

If a packet of length \( L \) is transmitted over a memoryless binary symmetric communication channel with average probability of data bit success \( Q_e \) and if the packet includes block error control capability that can correct 1 or fewer errors, then simple combinatorial arguments give the probability of packet success \( Q_E = g(x; L, t) \) as

\[
g(x; L, t) = \sum_{i=0}^{L-1} \binom{L}{i} (1-x)^i (x)^{L-i} \tag{6}\.
\]

By conditioning on the differential delays and phases between the desired and all interfering signals, and by assuming that \( N \) is large enough that interfering chips which overlap a desired bit boundary have negligible effect on \( Q_e \), we gain conditional independence of \( Q_e \) from data bit to data bit within the desired packet. This corresponds to a situation where all transmitters are stable enough that the \( s_k \) and \( \phi_k \) are selected from distributions uniform on \([0,1]\) and \([0,2\pi]\), respectively, at the start of the transmission of a desired packet, and these quantities then remain constant throughout the remainder of the packet. Now the probability of packet success may be found by averaging the conditional packet success probability over all possible values of \( s_k \) and \( \phi_k \) in a manner similar to that shown in [1], producing

\[
\bar{Q}_E = \int_0^\infty \Phi \left( \frac{N}{\sqrt{2\pi}}; L, t \right) f_\Phi(x) \, dx , \tag{7}
\]

where \( \Phi(x) = 1 - Q(x) \). Since this method uses the improved Gaussian approximation to obtain accurate probability of data bit values and accounts for bit-to-bit error dependencies within the packet, we will call this the “IGA-D” technique. Note also that \( \bar{Q}_E \) varies with the number of simultaneous users \( K \) which is reflected in the density \( f_\Phi(x) \); thus \( \bar{Q}_E(K) \) may be expressed as a function of \( K \).

To determine the effect that bit-to-bit error dependence has on packet performance, we can compare the results given by (7) to a technique that uses (5) to obtain an accurate approximation to the probability of data bit error while ignoring these error dependencies; we call this the “IGA” technique, and its corresponding packet success probability is

\[
\bar{Q}_E^{(1)}(K) = g(1 - \hat{P}_e^{(2)}(K); L, t) . \tag{8}
\]

In the following section, we will use the IGA-D procedure as a basis for evaluating the accuracy of the IGA and two other methods which are commonly used for finding packet success probabilities from which network throughput may be determined. For example, a worst-case lower bound on \( Q_E(K) \) may be found by setting \( s_k=0 \) and \( \phi_k=0 \) for all \( k \in \{2, \ldots, K\} \), and approximating this lower bound by

\[
\bar{Q}_E^{(2)}(K) = g\left( \Phi \left( \frac{N}{\sqrt{K-1}}; L, t \right) \right) . \tag{9}
\]

The computational complexity of (9), which we call the “WC” method, is very low because it involves evaluating only two simple functions of \( N \) and \( K \) for each \( K \). Another method for finding \( Q_E(K) \) is to use a standard Gaussian approximation (“SGA”) to the probability of data bit error, which by its nature ignores the effect of bit-to-bit error dependence in the packets. This method also has low computational complexity and uses (4) to produce

\[
\bar{Q}_E^{(3)}(K) = g(1 - \hat{P}_e^{(1)}(K); L, t) . \tag{10}
\]

Now we are prepared to develop an infinite-user slotted ALOHA network model suitable for use in DS/SSMA communications, from which we can determine packet throughput and make various comparisons.
IV. Infinite-User Slotted ALOHA Network Throughput

In this section we analyze the throughput of a simple infinite-user slotted ALOHA network to gain insight into the effect that using packet success probability calculations with varying degrees of accuracy has on network performance. First, we show the network throughput equations from which arbitrarily tight throughput bounds may be calculated when given $Q_E(K)$ for various values of $K$. We then work mainly with the lower throughput bound and quantify the performance enhancement gained by incorporating block error correction capability into the packets. The required redundancy of the error control code reduces the number of message bits in a fixed-length packet; we take this into account by defining and calculating a quantity called the "effective throughput." Next, we define network capacity as the maximum effective throughput, and we compare the capacity of the infinite-user slotted ALOHA DS/SSMA packet network with the capacity of a number of narrow-band networks using an equivalent bandwidth.

A. Packet Throughput Calculations

Consider a number of independent users (nodes) sharing a common communication channel and generating packets of length $L$ bits. The communication channel is assumed to receive newly-generated packets modeled as a Poisson process with rate $S$. The nodes employ a slotted ALOHA protocol, so when a packet arrives at a particular node, transmission will commence at the start of the next time slot. Each slot is long enough for a packet to be transmitted, and any associated guard time between slots is assumed short compared to the packet transmission time. It is also assumed that propagation delays and slot timing errors are such that the bit and packet success probabilities developed in the previous sections hold. From the network point-of-view, the channel receives newly-generated packets at rate $S$ and previously unsuccessful packets at rate $R$, also assumed Poisson. The offered channel traffic is then Poisson with rate $G = S + R$, from which we assume that the channel is stable, meaning that the throughput rate is also $S$ and that all newly-generated packets will be successfully transmitted within a finite time period. By defining a unit of time to be the duration of a slot, $G$ and $S$ may be specified in units of "packets per slot." The number of simultaneous users $K \in \{0, 1, \ldots\}$ in a particular slot is now a random variable with probabilities determined by $G$.

By following the techniques developed in [7], the throughput of a DS/SSMA slotted ALOHA packet network can be expressed as

$$S = Ge^{-G} + Ge^{-G} \sum_{k=1}^{\infty} \frac{G^{-k}}{k} Q_E(k+1). \quad (11)$$

The term $Ge^{-G}$ represents the throughput given that there is at most one packet in a slot; this is identical to the throughput of a narrow-band slotted ALOHA channel. The remaining terms in (11) represent the additional throughput realized by using DS/SSMA techniques.

An exact evaluation of (11) is not possible because of the infinite sum, but we can use the fact that $Q_E(k)$ is a decreasing function of $k$ to obtain arbitrarily tight upper and lower bounds on $S$. For example, a lower bound on the series may be found by simply truncating it after $Q_E(k+1)$ reaches some threshold; call it $Q_E(K_t)$:
Similarly, an upper bound on $S$ is established by letting $Q_E(k) = Q_E(K_u)$ for all $k > K_u$ and following the procedure in [6]. To avoid excessively busy plots, we will show only $S_i$ with $Q_E(K_u) \leq 10^{-3}$ for the throughput results in this paper.

We can now compare the WC, SGA, IGA, and IGA-D methods of calculating the probability of packet success in the network throughput equation when there is no error correction capability in the packets (Fig. 2). Convexity arguments in [1] show that the IGA method lower bounds packet success probabilities compared to the IGA-D method when no error control is used, and that the SGA approach is optimistic when the number of simultaneous users is small and pessimistic when $K$ is large; these characteristics are easily seen in Fig. 2. We also note that the WC technique produces a very loose lower bound on network throughput.

**B. Packet Error Control and Network Throughput**

In traditional narrow-band packet system analysis [8], packets are usually assumed to possess some form of error control so that only error-free data is accepted by the receiving node. Error detection allows the receiving node to identify packets which have been involved in a collision and reject them accordingly. Packets not involved in a collision are considered to have been sent over a noiseless channel and therefore will always be received correctly. Conversely, the communication channel is considered useless during the time that two or more narrow-band transmitters operate simultaneously; consequently, no amount of error correction capability will recover the affected packets, which are assumed lost.

Packets in the DS/SSMA network environment are not subject to such a catastrophic channel degradation, however, and will probably benefit from some form of error control that can correct the occasional data bit error caused by the multiple-access interference. However, the redundancy added by the error control code means that each packet of length $L$ has fewer than $L$ information bits in it, causing a corresponding reduction in actual data throughput. Suppose, for example, that $M$ message bits are to be encoded into a packet of length $L$ with a block error control code capable of correcting $t$ errors. Then the quantity $L-M$ represents the redundancy added to the information to obtain this error control capability.

If we let $R_c$ be the rate of the error control code, then we can define the effective throughput $T = SR_c$ as the number of equivalent packets of $L$ message bits required to obtain the same information throughput as $S$ packets of $M$ message bits and $L-M$ additional bits for error control. To produce a lower bound on $T$ as a function of $G$ for a given packet length $L$, signature sequence length $N$, and error correction capability $t$ based upon minimum distance $d_m=2t+1$, we use the Varsharmov-Gilbert lower bound on block error control code performance to obtain the result [6]

$$T \geq S_i \left[1+\tilde{A}\log_2 \tilde{A} + (1-\tilde{A})\log_2(1-\tilde{A})\right]$$

where

$$\tilde{A} = \frac{2t+1}{L}. \quad (14)$$

At this point we are not yet certain whether the addition of error correcting capability to the packets will increase or decrease the effective throughput of the network. Increasing $t$ will increase the average number of successful packets in a slot given
that $K$ packets are transmitted, but each packet now contains fewer information bits. Figure 3 shows a comparison of the lower bound on $T$, which we call $T_I$, vs $G$ for various values of $t$. (For $t=0$ we set $T_I = S_T$.) Note that as $t$ increases, the peak effective throughput exceeds that of a system with larger $N$ and no error correction capability, but without the additional bandwidth penalty incurred by increasing $N$. However, since the addition of error control reduces the portion of a packet which is devoted to the message, we see that as $t$ increases for a given $N$, the effective throughput curve peaks more slowly. For example, if $N=31$ and $G=5$, then a network of packets using $t=5$ produces a higher effective throughput than that given by the more-powerful $t=20$ error control code. As a consequence, for a given offered rate, increasing the error correcting capability for each node in the network may reduce the effective throughput. As the offered rate increases, however, the situation eventually reverses and the more powerful error control code produces the greater effective throughput.

C. Network Capacity

When designing a packet communication system, determining network capacity (maximum effective throughput) is an important consideration to avoid instability from throughput requirements which exceed the traffic-handling capability of the network. For notational consistency, we define network capacity $T_C = \sup T_I$, where the supremum is taken over the offered rate $G$. Plots of $T_C$ as a function of $N$ for the WC, SGA, IGA, and IGA-D analysis methods are given in Figs. 4 and 5. From the results shown in Fig. 3, we expect that when $t=0$ the IGA method will produce a lower-bound on $T_C$ for all $N$. As $N$ increases with $t>0$, the SGA and IGA methods generate results that are slightly optimistic compared to those from the more-accurate IGA-D approach. On the other hand, the WC method yields a very pessimistic assessment of capacity.

It is also interesting to examine the effect that error control alone has on network capacity. One of the main characteristics of error control is that added throughput is realized by increasing the probability that a packet is transmitted successfully for a given $N$ and hence for a given signal bandwidth. When the error control is powerful enough, the capacity per unit bandwidth of a DS/SSMA network compares favorably to that of a narrow-band slotted ALOHA system, which we now show.

D. Throughput/Bandwidth Comparisons

The advantages of using spread-spectrum in a packet radio system must be compared to the cost of this signaling technique in terms of increased bandwidth. To do this, we normalize the throughput by the signal bandwidth. This will enable us to compare the performance of a single DS/SSMA packet network to a number of narrow-band networks encompassing the same total bandwidth. To avoid arbitrary cutoff limits on the theoretically infinite bandwidths of the signals in question, we note that $N$, the number of chips per data bit in the spectral-spreading signal, is also approximately the ratio of the bandwidth of the spread-spectrum signal to that of the narrow-band signal.

Since the capacity of a narrow-band slotted ALOHA network is $1/e$ [8], we can define the \textit{capacity/bandwidth factor} $\beta$ as

$$\beta = \frac{e T_C}{N}$$

which is the ratio of the maximum effective throughput of the DS/SSMA packet system to...
the capacity of \( N \) narrow-band slotted ALOHA packet networks. Using the results derived in the previous sections, we can find \( \beta \) for various values of \( N \) and \( t \) (Fig. 6). It is apparent that a slotted ALOHA DS/SSMA packet system with no error control makes poorer use of channel bandwidth at capacity than using \( N \) narrow-band slotted ALOHA systems. However, when a sufficient amount of error control is employed to correct some of the packet errors caused by multiple-access interference, the network capacity increases beyond that of \( N \) narrow-band systems. Because of our assumption of a useless channel during collisions, narrow-band slotted ALOHA systems without some form of signal capture capability will not realize any additional throughput from error correction.

V. Conclusions

Using the improved Gaussian approximation for the probability of data bit error in a DS/SSMA radio system offers accurate figures along with a reduction in computational complexity over previous attempts to improve the accuracy of the standard Gaussian approximation. The conditioning performed on relative delays and phases of the interfering signals also allows us to calculate accurate packet throughput values by incorporating the effects of bit-to-bit error dependence on each packet in the network. When all users employ signature sequences that are completely random, the simple and widely-used standard Gaussian approximation produces packet success and throughput results that are easy to calculate and reasonably accurate. Conversely, assuming a worst-case chip and phase alignment situation between all signals and taking a Gaussian approximation to the probability of data bit error under these conditions produces results which are very pessimistic compared to system performance modeled by the improved Gaussian approximation incorporating the effect of bit-to-bit error dependence.

The network capacity, which we define as the maximum effective throughput, is an important design criterion. For example, if we insure that the network capacity is significantly higher than the desired throughput, then the retransmission rate associated with failed packets becomes less important to network stability. We compared the capacity of the DS/SSMA packet network to that of \( N \) narrow-band slotted ALOHA systems which occupy approximately the same bandwidth. When no error control is used, the DS/SSMA system is only about half as efficient as the relatively inefficient slotted ALOHA signaling technique, but as the error control code becomes more powerful, the capacity of DS/SSMA eventually exceeds that of \( N \) narrow-band networks.
References


Figure 1. Bounds on the probability of data error vs number of simultaneous users.

Figure 2. Throughput vs offered rate for various analysis techniques (N=31, L=1000, t=0).

Figure 3. Effective throughput vs offered rate (SGA).

Figure 4. Network capacity vs signature sequence length (L=1000, t=0).

Figure 5. Network capacity vs signature sequence length (L=1000, t=10).

Figure 6. Capacity/bandwidth factor (IGA-D) vs number of correctable errors (L=1000).