pression for the \( \hat{C}_q \) coefficient is given by

\[
\text{Var} \hat{C}_q \geq \frac{N_0}{A^2 T} \left( \frac{2}{T} \right)^2 \frac{1}{2q+1} \left( \frac{s+q+1}{2} \right) \left( \frac{s-q-1}{2} \right) q!
\]

where

\[
s = \begin{cases} 
p, & p + q = \text{odd} \\
1 + p, & p + q = \text{even}. \end{cases}
\]

Of special practical interest are \( \text{Var} \hat{C}_0 \) and \( \text{Var} \hat{C}_1 \) which correspond to delay and Doppler, respectively. Setting \( q = 0 \) and \( q = 1 \) in (29), we obtain

\[
\text{Var} \hat{C}_0 \geq \frac{N_0}{A^2 T} \left( \frac{2}{T} \right)^2 \left( \frac{s}{2} \right)^2 \frac{1}{2} \]

where

\[
s = \begin{cases} 
p, & p \text{ odd} \\
1 + p, & p \text{ even}. \end{cases}
\]

and

\[
\text{Var} \hat{C}_1 \geq \frac{N_0}{A^2 T} \left( \frac{2}{T} \right)^2 \frac{(s+1)!}{2^{s+1}} \]

where

\[
s = \begin{cases} 
p + 1, & p \text{ odd} \\
p, & p \text{ even}. \end{cases}
\]

When the motion can be represented by a fixed range, the smallest polynomial for an unbiased estimate of the delay is of the zero order. Thus, from (31) and (32) we obtain

\[
\text{Var} \hat{C}_0 \bigg|_{p=0} \geq \frac{N_0}{A^2 T}.
\]

Note that increasing the polynomial order from zero to one does not effect the variance \( 0 \).

When the motion results in range rate (Doppler), the smallest polynomial allowing its estimate is of the first order. Thus, from (33) and (34) we obtain

\[
\text{Var} \hat{C}_1 \bigg|_{p=1} \geq \frac{N_0}{A^2 T} \frac{12}{T^2}.
\]

Both (35) and (36) are well-known results [6].

IV. ASYMPTOTIC EXPRESSIONS

When unbiased estimates of delay and Doppler are calculated simultaneously with many higher derivatives, their variances [(31) and (33)] approach the asymptotic expressions

\[
\text{Var} \hat{C}_0 \bigg|_{s \gg 1} \geq \frac{N_0}{A^2 T} \frac{2s}{\pi}
\]

where \( s \) is related to \( p \) as in (32), and

\[
\text{Var} \hat{C}_1 \bigg|_{s \gg 1} \geq \frac{N_0}{A^2 T} \frac{2s^2(s+1)}{3\pi}
\]

where \( s \) is related to \( p \) as in (34).

Equations (37) and (38) provide an answer to the question left open in [1]: How do the variances of delay and Doppler increase with the addition of arbitrary large polynomial order? Equations (37) and (38) show (asymptotically) that the delay variance increases linearly with the polynomial order, and the Doppler variance increases cubically.

REFERENCES


A Two-Parameter Class of Bessel Weightings for Spectral Analysis or Array Processing—The Ideal Weighting-Window Pairs

ALBERT H. NUTTALL

Abstract—A unified theory for array processing in 1, 2, or 3 dimensions is pointed out and illustrated by a two-parameter class of Bessel weightings. This class subsumes the I_s-sinh weighting-window pair as well as the ideal space factor of van der Maas in one dimension. The weightings that realize the ideal space factor in 2 and 3 dimensions are generalized functions more singular than the delta function required in 1 dimension.

I. INTRODUCTION

A wide variety of time-domain weightings for spectral analysis, whose frequency-domain windows have very good sidelobe behavior, are available in [1], [2]. However, most of the weighting-window pairs in [1], [2] have no parameters in their design equations; that is, the windows are fixed and cannot be altered, as for example, in the Hanning and Hamming windows. A few windows, such as the Dolph–Chebyshev and I_s-sinh [3], [4], do have a single parameter in their design equations that allows for a tradeoff between the mainlobe width and the ratio of mainlobe-to-peak-sidelobe. However, the latter have no control over the rate of decay of the sidelobes, the Dolph–Chebyshev case having no decay, and the Kaiser–Bessel case a 6 dB/octave decay. It is obvious that in order to control both

Manuscript received September 27, 1982; revised May 3, 1983.

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TABLE I
DEFINITIONS OF $\mu$ AND $u$

<table>
<thead>
<tr>
<th>Number of Dimensions</th>
<th>Value of $\mu$</th>
<th>$u$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{2}$</td>
<td>$2\pi R \left(\sin \phi_d - \sin \phi_s\right)$ or $2\pi f$</td>
</tr>
<tr>
<td>2</td>
<td>0</td>
<td>$2\pi R \left(\sin^2 \phi + \sin^2 \phi_d - 2 \sin \phi \sin \phi_d \cos (\phi - \phi_s)\right)^{1/2}$</td>
</tr>
<tr>
<td>3</td>
<td>$\frac{1}{2}$</td>
<td>$2\pi R \left(1 - 2 \cos \phi \cos \phi_d - 2 \sin \phi \sin \phi_d \cos (\phi - \phi_s)\right)^{1/2}$</td>
</tr>
</tbody>
</table>

For spectral analysis where the temporal weighting is zero for $|t| > L$ seconds, we have $u = 2\pi f$, where $f$ is frequency in Hz.

the sidelobe decay and the mainlobe-to-peak-sidelobe ratio, a two-parameter family of weightings is necessary.

In the antenna field, a two-parameter family of distributions was proposed in 1956 for a linear aperture [5] and seems to have been overlooked by the signal processing community. In fact, this important family subsumes the $L_{-\alpha}$-sinh window; indeed, this latter window, popularly known as Kaiser-Bessel, is originally due to Taylor [6] in 1953. Furthermore, a limiting operation [7] on the two-parameter family yields the ideal space factor of van der Maas [8]. We show here that this two-parameter class has applications for array processing in 1, 2, or 3 dimensions, and that the ideal space factor, which requires generalized functions for its realization, is a limiting member of the class. The results in one dimension have immediate application to spectral analysis, since a Fourier transform connects the window (amplitude pattern) with the weighting (aperture distribution).

II. UNIFIED FORM FOR RESPONSES OF ANTENNAS IN DIFFERENT DIMENSIONS

The far-field amplitude pattern of an antenna with an isotropic element factor is the Fourier transform of the aperture distribution [9], [10]. This has led to the observation that, for linear and circularly symmetric planar antennas, the linear and circular apertures have a common theory [10], [11]. This is in fact true for 3 dimensions as well, for a spherically symmetric volumetric antenna [7]. Specifically, the far-field amplitude pattern is given by ([12] in [7])

$$g(u) = \int_{0}^{1} ds \frac{s^\mu}{\sqrt{\sin \theta}} J_\mu \left(\frac{R u}{\sin \theta}\right) w(s)$$

(1)

where $s = 0$ and 1 correspond to the center and edge of the antenna, respectively, $w$ is the aperture distribution, and Table I delineates the choices of parameters $\mu$ and $u$. Here $R$ is the radius of the antenna, $\lambda$ is the wavelength of a single-frequency plane wave arriving at (polar, azimuthal) angle $(\phi_d, \theta_d)$, and $(\phi_l, \theta_l)$ is the look angle of the time-delay steered antenna. We shall refer to aperture distribution $w$ as the weighting and the far-field amplitude pattern $g$ as the window; these are weightings in space and windows in spatial wave number.

III. THE TWO-PARAMETER CLASS OF BESSEL WEIGHTINGS

The two-parameter weighting of interest is

$$w(s) = \left(\frac{\sqrt{1 - s^2}}{B}\right)^\mu J_\mu \left(B \sqrt{1 - s^2}\right) \quad \text{for} \quad 0 \leq s \leq 1$$

(2)

where $\nu$ and $B$ are real constants. Substitution of (2) in (1) yields window [7]

$$g(u) = J_{\mu + \nu + 1}(\sqrt{u^2 - B^2})$$

$$= \ell_{\mu + \nu + 1}(\sqrt{B^2 - u^2}) \quad \text{for all} \quad u$$

(3)

where we define

$$J_\alpha \equiv J_\alpha(z)/z^\alpha, \quad \ell_\alpha \equiv I_\alpha(z)/z^\alpha.$$  

(4)

These latter functions are closely related to the Lambda functions [10], [11]. The asymptotic decay of the window is $6\mu + 6\nu + 9 \text{ dB/octave}$, for any $B$.  

(5)
Thus each additional dimension adds 3 dB/octave to the decay if $v$ is unchanged; see also [12, p. 20].

When $\mu + v + 1 = -0.5$, the window in (3) becomes $\sqrt{2/\pi} \cdot \cos(\sqrt{u^2 - B^2})$, which is the ideal space factor for an antenna [8]-[10], [12]. It has equal sidelobes and a peak-to-sidelobe voltage ratio $\cosh(B)$. A plot of the first null location of $g(u)$ versus $B$ is given in Fig. 1 for general $\mu$ and $v$; it applies to any number of dimensions, when $a$ is interpreted as $\mu + v + 1$. The corresponding peak sidelobe level is given in Fig. 2. The fundamental trade-offs between peak sidelobe and first null location for various decay rates are depicted in Fig. 3. The bottommost curve in Fig. 3 is the ideal space factor; no antenna can perform to the left of this curve.

### IV. REQUIRED WEIGHTING FOR IDEAL SPACE FACTOR

In order to realize the ideal space factor $\sqrt{2/\pi} \cos(\sqrt{u^2 - B^2})$, the weighting $w(s)$ must contain generalized functions. In particular, we have the results [7, pp. 14-15] in Table II where $r = \sqrt{1-s^2}$. The delta function required in one dimension is well known; for example, see [10, p. 9-26]. The results for 2 and 3 dimensions are new; in particular, the $I_{-3/2}$ result fills the blank in [10, p. 9-26, Table 4].

Whereas the singularities in 1 and 3 dimensions are conspicuous in Table II, namely $\delta$ and $\delta'$ the function $I_{-3/2}$ is a generalized function that is singular at $s = 1$. Approximations to these generalized functions are developed in detail in [7], [13].

### TABLE II

<table>
<thead>
<tr>
<th>Number of dimensions</th>
<th>Required Weighting</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{B}{r} I_1(Br) + \delta(s - 1)$</td>
</tr>
<tr>
<td>2</td>
<td>$\left(\frac{B}{r}\right)^{3/2} I_{3/2}(Br)$</td>
</tr>
<tr>
<td>3</td>
<td>$\left(\frac{B}{r}\right)^2 I_2(Br) + \left(\frac{B^2}{2} - 1\right) \delta(s) - \delta'(s - 1)$</td>
</tr>
</tbody>
</table>

Here $r = \sqrt{1-s^2}$.

### REFERENCES


A Fast Convergence Frequency Domain Adaptive Filter

JUAN CARLOS OUGÜE, TSUNEHO SAITO, AND YUKIO HOSHIKO

Abstract—This correspondence presents a new fast convergence algorithm for frequency domain adaptive filter and its applicability to acoustic noise cancellation in speech signals.

I. INTRODUCTION

Adaptive digital filters have many application areas in signal processing such as noise or echo cancelling, signal enhancement, linear prediction, automatic channel equalization, etc. [1]-[3]. The most familiar structure of an adaptive filter is the transversal one implemented as a finite impulse response (FIR) filter. The filter weights are updated by the adaptive algorithm to minimize a determined performance function. Normally, the most used performance function is the mean square error (MSE).

Since Widrow and Hoff published the least-mean-square (LMS) algorithm in 1960, research on adaptive digital filters has been of continuing interest to scientists and engineers. The LMS algorithm is an application of the steepest descent method, and the correction parameter \( \mu \), which controls the adaptation of the filter weights, is a constant factor to be held very small in order to assure stability. Due to this condition the convergence speed of the LMS algorithm is slow.

Adaptive filters are used in situations where the statistics of the input signals are unknown or variable in time. In such a nonstationary environment, fast convergence properties are inevitably required. In some cases, the conventional LMS algorithm converges too slowly and is unable to follow fast changes in the input signal characteristics.

Next, Godard and Ahmed proposed different fast convergence algorithms obtained by orthogonalizing the input signal [4], [5]. However, these algorithms require tedious matrix operation, and when the filter length is large, the computational requirement is enormous.

On the other hand, it was shown that computational saving may be obtained for large filter length implementing the LMS algorithm in the frequency domain, due to the efficient fast Fourier transform (FFT) algorithm [6]-[9].

In this paper, a new fast convergence algorithm for adaptive digital filters implemented in the frequency domain is proposed. The proposed algorithm shows the same convergence behavior as Godard’s algorithm, but requires substantially fewer computations and may therefore be considered as a viable alternative for real-time processing.

II. FREQUENCY DOMAIN ADAPTIVE DIGITAL FILTERS

A. Filter Structure

The frequency domain adaptive digital filters are implemented as illustrated in Fig. 1, by sectioning the input signal \( x_n \) and the desired response \( d_n \) to form \( N \)-point data blocks. They are then transformed to the frequency domain by the \( N \)-point discrete Fourier transform (DFT) [6].

Let \( X_k(m) \) and \( D_k(m) \) be the \( k \)th frequency bin in the \( m \)th data block of the input signal and desired response, respectively. There are \( N \) complex weights \( W_k(m) \), one corresponding to each frequency bin. The weighted outputs are given by

\[
y_k(m) = W_k(m) X_k(m)
\]

and are fed to an inverse FFT operator to produce \( N \)-point output signals.

The weighted outputs are subtracted from the desired response transform values at corresponding frequencies to form \( N \) complex error signals:

\[
e_k(m) = D_k(m) - y_k(m).
\]

If the input signals are wide-sense stationary over the observation time, then the frequency components become orthogonal and each of the \( N \) complex weights may be updated independently. Therefore, the frequency domain adaptive filters may be treated as composed of \( N \) single weight adaptive filters.

B. A Fast Convergence Algorithm

A fast convergence algorithm for frequency domain adaptive filters is proposed in this section. The weights update equation is as follows:

\[
W_k(m + 1) = W_k(m) + P_k(m) X_k^*(m) e_k(m)
\]

with

\[
P_k(m) = [P_k^{-1}(m - 1) + X_k^*(m) X_k(m)]^{-1}.
\]

A variable coefficient \( P_k(m) \) controls the convergence rate and, because it was made dependent on the input signal, a fast convergence feature is expected. * indicates complex conjugate.

Equation (4) may be rewritten as

\[
P_k(m) = \left[ P_k^{-1}(0) + \sum_{i=1}^{m} X_k^*(i) X_k(i) \right]^{-1}.
\]

Substituting (1) and (2) into (3), we have

\[
W_k(m + 1) = W_k(m) + P_k(m) X_k^*(m) D_k(m) - P_k(m) W_k(m) X_k^*(m) X_k(m).
\]