Acoustic Transfer Function of the Ocean for a Motional Source

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Abstract—The motional-source transfer characteristics of long-range acoustic propagations are derived based on the discrete multipath propagation model predicted by geometrical acoustics. Parameters are those associated with the characteristic solutions of the eikonal equation. A modified transfer function is formulated which relates the received signal to a particularly useful transformation of the source signal. The modified transfer function shows that a remotely received signal consists of the weighted superposition of a number of modified source signals both slightly compressed (or expanded) and translated in time. The time variable compression factor is a stochastic process which reflects the fluctuations inherent in both the medium and the source motion. Application of the model transfer function to a specific ocean profile exemplifies the ocean filter characteristics and the complex nature of the received signal from a CW source in motion. It is concluded that the model transfer function will be a useful tool for signal processing applications in underwater acoustics.

INTRODUCTION

TO HELP understand signal distortion and coherence degradation in long-range propagations, the ocean can be conceived as an acoustic communications channel. This concept is portrayed in Fig. 1, where the transfer-function block represents the ocean medium (including its boundaries) between the moving source and a fixed receiving sensor (or array). The symbolic terms $H(\omega; t)$ and $h(t, t_0)$ represent the transfer function and the impulse response of the system, respectively. In the above representation the ocean medium is assumed to be linear but can be time variable.

The development of a precise deterministic transfer function for a real ocean system can never be realized in practice. However, it is helpful to have a model transfer function in order to understand and to assess the influence of the medium, and the source signal and motional parameters, on the remotely received signal. The present paper addresses the problem of modeling the motional-source ocean transfer function based upon the discrete multipath arrival structure predicted by geometrical acoustics [1]. Parameters in the model are those associated with the characteristic solutions of the eikonal equation [1], [2]. Specific solutions to the distortion induced by multipath propagation, for particular idealizations of the ocean profile and source signal and motion, are available [3], [6]. The present paper represents a more general approach to the problem where the solution is dependent on only the eigenray parameters (initial ray angle, time delay, and propagation loss) for a dispersionless medium.

THE MULTIPARTITE SIGNAL MODEL

The ocean medium is known to behave as a form of acoustic lens (in the vertical plane), wherein acoustic rays emanating from a source are continuously being refracted (or boundary reflected and refracted) as the pressure wave propagates away from the source [1]. Given the system geometry and medium characteristics (sound velocity profiles, etc.) only sound rays emitted at particular depression (or elevation) angles will arrive at the receiving sensor, and the discrete eigenray (multipath) parameters (initial ray angle, delay time, and propagation loss) may be computed by means of a ray-trace program. Thus, the received signal $r(t)$ will be the superposition of a number of discrete signals arriving over different paths, and the acoustic transfer function (Fig. 1) can be constructed as a number of eigenray transfer functions operating in parallel. The problem of determining the ocean transfer function is thereby reduced to the task of determining the transfer function for each of the eigenray paths and summing the results from all of the relevant paths (Fig. 2).

THE EIGENRAY TRANSFER FUNCTION

For convenience, each eigenray transfer function can be treated as the cascade of a minimum-phase system and an all-pass system (Fig. 3). The all-pass function represents only the time delay of the signal propagation, while the minimum-phase function reflects the transfer characteristics which influence the amplitude of the received signal. The physical realizability of the minimum-phase function is not of concern since it is known that the composite transfer function is physically realizable.

All-Pass Transfer Function

To determine the all-pass function it is sufficient to determine the impulse response of the system; or equivalently, the propagation time between the source and the receiving sensor as a function of time. The propagation time will first be computed relative to the time an impulse is emitted from the source. In the analyses to follow it will be assumed that the distance between the moving source and the fixed receiving sensor is very large compared with the source movement over the temporal analysis interval of interest.

The Basic Equations: In general, the propagation time $\tau$ for an impulse leaving the source at time $t$ can be written as

$$\tau = \tau(R, z; t) = L(R, z; t) \bar{v}(R, z; t)$$

where $L(R, z; t)$ is the length of the propagation path and $\bar{v}(R, z; t)$ is the average sound propagation velocity over the path. The symbol $R$ is the horizontal (great circle) range be-

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As one might expect, the factor in parentheses is minus the component of source velocity along the relevant eigenray path at the point of source.

It should be evident that in general, and for a dispersionless medium,

$$\frac{d\tau}{dt} = - \frac{V \cdot \mathbf{I}_e}{c(R, z, t)} = \frac{\dot{R} \cos \varphi + \dot{z} \sin \varphi}{c(R, z, t)}$$ (1)

where $V \cdot \mathbf{I}_e$ is the inner or dot product of the source velocity vector and a unit vector along the relevant eigenray path at the source, $c(R, z, t)$ is the sound propagation speed at the source, and $\varphi$ is the elevation angle of the unit vector. In the practical situation all three parameters ($V$, $\mathbf{I}_e$, and $c$) can be expected to fluctuate slowly with time. In addition, there exists other medium dynamics which induce fluctuations in the rate of change of $\tau$. For purposes of analyses, all of the fluctuations will be combined into a single variable which is representative of the random $\dot{\tau}$ fluctuations to be expected.

**Fluctuation Parameters:*** Except for unusual conditions it can be expected that the $\dot{\tau}$ fluctuations will be dominated by the speed and course fluctuations inherent in the source motion. Although the vertical fluctuations of both the source and sea-surface motion may be comparable to these, their influence on $\dot{\tau}$ is moderated by the acuteness of the eigenray angle $\varphi$. (For long-range propagation, the eigenray angle $\varphi$ will generally range within about $\pm 15$ degrees.) Further, the medium induced fluctuations in $\dot{\tau}$ can be transformed into equivalent source motional fluctuations which are additive to the true motional fluctuations. In any event, taking into account only the horizontal motional fluctuations of the source will provide a valid and representative measure of the influence of both the medium and the motional fluctuations on $\dot{\tau}$.

Consider then a source which is in horizontal motion at a speed $v$ and along a base course $\Psi$ relative to the range axis $R$. Over the analysis interval $T$ let

$$v = v_0 + a_v \xi_1$$

and

$$\Psi = \Psi_0 + a_\Psi \xi_2$$

where $v_0$ and $\Psi_0$ are the mean values, $a_v$ and $a_\Psi$ are the standard deviations, and $\xi_1$ and $\xi_2$ are zero-mean variables with unity variance. For $a_\Psi \xi_2 \ll 1$, $a_v \xi_1 \ll v_0$, and $\xi_1$ and $\xi_2$ uncorrelated, it can be shown that (1) approximates

$$\frac{d\tau}{dt} = \frac{k_0}{c_0} - 1 - e \bar{\xi}$$

(2a)

where

$$k_0 = \left[ 1 - \frac{v_0}{c} \cos \Psi_0 \cos \varphi \right]^{-1}$$

(2b)

$$e = \frac{\bar{\sigma}_v}{c} \cos \varphi$$

(2c)

$$\bar{\sigma}_v = \sqrt{(a_v \cos \Psi_0)^2 + (v_0 a_\Psi \sin \Psi_0)^2}$$

(2d)
The zero-mean and unity variance variable $\xi$ is intended to reflect the effect of all the system fluctuations on the rate of propagation time. The parameter $\epsilon$ is a constant representing the standard deviation of the $\tau$ fluctuations. (In reality this parameter is more inclusive than that shown; however, only the more dominant terms have been retained.) The parameter $k_0$ is a time scale-factor which reflects the mean time-compression (or expansion) of the source signal due to the system dynamics. This parameter may also be written as $1 + \delta_0$ where $\delta_0 (\delta_0 \ll 1)$ is the time scale-factor shift or Doppler ratio [7].

Propagation Time: Integrating (2a) gives the propagation time

$$\tau(x) = \tau_0 + (k_0^{-1} - 1)x - \epsilon \int_0^x \xi \, dt.$$  

This equation gives the propagation time in terms of the time $x$ that a signal event is emitted from the source. The time $\tau_0 = \tau(0)$ is the initial time delay at the start of the analysis interval $T$ ($0 \leq x \leq T$). The expression approximates a linear curve with small undulations about the mean slope due to the slowly varying medium and source-motion fluctuations. It will prove convenient to write the propagation time as

$$\tau(x) = \tau_0 + (k^{-1} - 1)x$$

where

$$k^{-1}(x) = 1/k(x) = k_0^{-1} - \frac{\epsilon}{x} \int_0^x \xi \, dt.$$  

In the above form, the parameter $k$ reflects the time scale-factor (signal-time transformation) resulting from both the mean and fluctuating dynamics [7]. This time variable parameter satisfies the differential equation

$$\frac{d}{dx} (k^{-1}x) = 1 - \frac{V \cdot I_\phi}{c} = 1 + \frac{d\tau}{dx}$$

whose solution is

$$k^{-1}x = x - \int_0^x \frac{V \cdot I_\phi}{c} \, dt = x + \tau(x) - \tau_0.$$  

As a consequence, the all-pass function transforms the temporal scale of the source signal into $x(t)$, where the time scale-factor function $k = 1/k^{-1}$ defines the minute time-compressions (or expansions) of the original time scale.

Propagation Time Fluctuations: Letting $\Delta \tau$ represent the deviation of $\tau(x)$ from linearity,

$$\Delta \tau = -\epsilon \int_0^x \xi \, dt = -\epsilon [\xi(x) + K]$$

where

$$\xi(x) = \int_0^x \xi \, dt - K$$

and where

$$K = \frac{1}{T} \int_0^T \left[ \int_0^x \xi \, dt \right] \, dx$$

is a state variable chosen to make $\xi$ a zero-mean variable over the analysis interval $T$. The mean and standard deviation of $\Delta \tau$ are $-\epsilon K$ and $\sigma_{\Delta \tau} = \epsilon \sqrt{\xi}$, respectively. Although the expected value (or ensemble average) of $K$ will be zero, over any given analysis interval $K$ can be finite. This implies that, over a given analysis interval, the initial time delay is virtually biased by an amount $-\epsilon K$ relative $\tau_0$. The value of $\sigma_{\Delta \tau}$ has been previously derived [8], so that the standard deviation of $\Delta \tau$ may be written as

$$\sigma_{\Delta \tau} = \sigma_{\Delta \tau} = \epsilon T \sqrt{\sum_1^n \frac{1}{n^2} P_\xi \left( \frac{n}{T} \right) \int_0^T \frac{P_\xi \left( \frac{n}{T} \right) \, dt}{2\pi}$$

where $P_\xi(f)$ is the power spectral density of $\xi$. The relation shows that the standard deviation $\sigma_{\Delta \tau}$ is maximum (and equal to $\epsilon T/2\pi$) when the total power in $\xi$ is concentrated at a frequency equal to $1/T$ Hz. When $\xi$ is a pure sinusoid equal to $\sqrt{2} \sin (2\pi x/T + \phi)$, the mean and standard deviation of $\Delta \tau$ become $(\sqrt{2} \epsilon T/2\pi) \cos \phi$ and $\epsilon T/2\pi$, respectively. For a 30 min analysis interval $eT/2\pi$ is approximately 0.1 $\sigma_{\Delta \tau}$. Consequently, over analysis intervals less than about 30 min and $\sigma_{\Delta \tau}$ less than one knot, the virtual bias and the fluctuations of
(about a linear curve) can amount to, at most, only a small fraction of a second.

**Signal-Time Transformation:** The propagation time $\tau$ given by (3) expresses the value of the variable in terms of the time $x$ that a signal event is emitted from the source. Since the time $t$ that the signal event arrives at the receiver is simply $x + \tau(x)$, the relationship between the two time frames of reference is (see Appendix)

$$t(x) = \tau_0 + k^{-1}x = \tau_0 + k_0^{-1}x - \varepsilon \int_0^x \xi \, dx$$  \hspace{1cm} (6a)

and

$$x(t) = k(t - \tau_0)$$  \hspace{1cm} (6b)

where

$$k \approx k_0 \left[1 - \frac{\varepsilon}{t - \tau_0} \int_0^{k_0(t - \tau_0)} \xi(x) \, dx\right]^{-1}.$$  \hspace{1cm} (6c)

The propagation time may therefore be written as [9]

$$\tau \{x(t)\} = t - k(t - \tau_0).$$  \hspace{1cm} (7)

The above expression gives the propagation time experienced by a source-signal event arriving at the receiving sensor at time $t$.

The signal-time transformation is depicted graphically in Fig. 5. (In the figure the time compression has been greatly exaggerated for purposes of demonstration.) In the practical case, the compression factor will have accordion-like fluctuations superimposed along the time axis, reflecting the effect of the motional and medium fluctuations. These temporal fluctuations are sufficiently minute and slowly varying so that they would not be perceived without the use of special processing equipment.

**All-Pass Transfer Function:** From (6) and (7) it is now possible to write the impulse response of the all-pass function as

$$h_A(t; t_0) = \delta \{t - \tau[x(t)]\} = \delta \{k(t - \tau_0)\}.$$  \hspace{1cm} (8)

where $\delta(-)$ is the Dirac delta function of the indicated argument and $t_0 \equiv x$. That is, an impulse arriving at the receiving sensor at time $t$ would have been emitted from the source at time $t - \tau[x(t)] = k(t - \tau_0)$. The all-pass transfer function may be obtained by taking the Fourier transform of the impulse response. Thus,

$$H_A(i\omega; t) = \int_{-\infty}^{\infty} e^{-i\omega \xi} \delta \{\xi - \tau[x(t)]\} \, d\xi,$$

$$= e^{-i\omega \xi_0 - (k - 1)(t - \tau_0)}.$$  \hspace{1cm} (9)

The all-pass transfer function reflects the phase characteristics of the received signal relative to those transmitted at the source.

**Minimum-Phase Transfer Function**

The minimum-phase function (Fig. 3) reflects the spectral amplitude characteristics experienced by the source signal in propagating over the eigenray path. The three causes of signal attenuation (or loss) are due to spreading, absorption, and boundary interaction. The latter two causes will be both frequency and range dependent, while the first will be only range dependent for a dispersionless medium.

**Spreading Loss:** Spreading loss results from the divergence of the sound pressure wave as it propagates away from the signal source. At long ranges the power loss due to horizontal spreading will be proportional to the range $R$. However, the loss due to vertical spreading is dependent on the refractive characteristics of the medium, and will be a complex function of the range and the source and receiving sensor depths. The signal attenuation due to spreading can be written as $\alpha(R; t)$ over which $\alpha(R; t)$ will be nonzero (for a given eigenray path) is rather restricted. This considerably limits the total number of eigenray paths which are significant in a practical problem.

**Absorption Loss:** The attenuation resulting from sound absorption is a function of both frequency and range. The acoustic absorption loss of the ocean has been studied rather extensively over the years [10], [11] and most recently the absorption coefficient has been found to approximate [11]

$$\gamma_0 + \frac{0.11Kf^2}{1 + f^2} + 0.011f^2 \text{ dB/km}$$

where $f$ is the signal frequency in kHz, and $\gamma_0$ and $K$ are constants which vary with the location in the ocean. Consequently, for frequencies less than about 300 Hz, the signal attenuation due to the medium absorption may be expressed as

$$e^{-(a_0 + a\omega^2)R}$$

where $a_0$ and $a$ can be expected to range from about $10^{-4}$ to $5 \times 10^{-4}$ and $2 \times 10^{-10}$ to $3 \times 10^{-10}$, respectively.

**Boundary Interaction Loss:** Probably the most complex attenuation an eigenray signal will experience is that achieved at an ocean boundary (sea surface or bottom). This loss is
the result of energy which may be absorbed into the boundary and/or irrecoverably scattered. In particular, bottom interactions are, more often than not, highly lossy. And there is no satisfactory way of analytically treating highly complex bottom interactions. However, it does not appear necessary to be too concerned about the highly complex cases. The received signal will be comprised of the sum of signals from a number of eigenray paths. And only the stronger of these signals will play a significant role in structuring the composite received signal. As a consequence, the expression chosen for the boundary interaction loss is based upon the quantitative formulation of the Rayleigh criterion [12]. This is

\[ s(t) = e^{-(\beta_0 + \beta_0 \omega^2)} \]

where \( \beta_0 \) is a constant and \( \beta = 2(\sigma c)^2 \sin^2 \Psi \), and where \( \sigma^2 \) is the variance of the boundary roughness, \( c \) is the sound propagation speed at the boundary, and \( \Psi \) is the grazing angle of the incident eigenray path. The first term of the exponent represents the loss due to absorption across the boundary. For surface interactions this loss will be negligible. The second term in the exponent represents the scattering (or reflection) coefficient in the specular direction. For a perfectly smooth boundary, this term would be zero. The negative sign in the relation applies for a surface reflection (or pressure release boundary) where the acoustic pressure is inverted after reflection.

For a total of \( N_r \) reflections along the eigenray path the attenuation due to the boundary interactions becomes

\[ (-1)^m e^{-(b_0 + b \omega^2)} \]

where

\[ b_0 = \sum_{j=1}^{N_r} \beta_0 j, \quad b = \sum_{j=1}^{N_r} \beta_j \]

and \( m \) is the number of surface reflections. Except under unusual circumstances, the eigenray signals of significance will be comprised of purely refraction (RR) and/or refraction surface-reflection (RSR) paths over long ranges in the deep ocean. In this event, \( N_r \) and \( m \) (for the RSR paths) will approximately equal the integer truncation of \( R/35 \) where \( R \) is the range in nmi.

**Minimum-Phase Function:** Combining the attenuation from all causes gives the minimum-phase function as

\[ H_m(i\omega; t) = (-1)^m \frac{\alpha(R; t)}{\sqrt{R}} e^{-(a_0 R + b_0) - (aR + b) \omega^2} \]

\[ = (-1)^m \frac{\alpha(R; t)}{\sqrt{R}} e^{-(a_0 R + b_0) - (\pi/4) (\omega/\omega_0)^2} \]  

(10a)

where

\[ f_i = \omega_f/2\pi = (\frac{1}{4})(1/\pi(aR + b)]^{1/2} \]  

(10b)

is the information bandwidth of the eigenray path [13]. The minimum-phase function for low frequencies (less than about 300 Hz) is therefore represented by a Gaussian shaped low-pass filter whose impulse response is

\[ h_m(t; t_0) = (-1)^m 2f_i \frac{\alpha(R_0; t_0)}{\sqrt{R_0}} e^{-(a_0 R_0 + b_0) - 4\pi(f_i t)^2} \]

(11)

where \( R_0 = R(t_0) \). The impulse response of a given eigenray path is therefore a Gaussian pulse with standard deviation equal to \( 1/(2\sqrt{2\pi f_i}) \approx 1/(5f_i) \). The above response is, of course, not physically realizable by itself. However, with the addition of the all-pass function, the composite response will be physically realizable for all practical purposes.

**Eigenray Transfer Characteristics**

The eigenray transfer function may be obtained by appropriately combining the minimum-phase and the all-pass functions. Thus,

\[ H_n(i\omega; t) = H_m(i\omega; k(t - \tau_{on}); R_0;i\omega; t) \]

\[ = A_n(i\omega; t) e^{-i\omega [\tau_{an} - (k_n - 1)(t - \tau_{on})]} \]

(12a)

where

\[ A_n(i\omega; t) = (-1)^m \frac{\alpha_n(R; x)}{\sqrt{R}} e^{-(a_0 R + b_0) - (\pi/4) (\omega/\omega_0 t)^2} \]

(12b)

and where the temporal argument of the variable parameters is taken as \( x = k_n (t - \tau_{an}) \).

The eigenray impulse response will be the convolution of the minimum-phase and the all-pass impulse functions. Thus,

\[ h_n(t; t_0) = \int_{-\infty}^{\infty} h_m(\xi; t_0) h_a(t, \xi; t_0) d\xi \]

\[ \approx B_n(x_0) e^{-4\pi f_i^2 (t - \tau_{on} + (k_n - 1) t_0)^2} \]

(13a)

where

\[ B_n(x_0) = (-1)^m 2f_i \frac{\alpha_n(R_0; x_0)}{\sqrt{R_0}} e^{-(a_0 R_0 + b_0)} \]

(13b)

and where the temporal argument \( x_0 \) of the variable parameters is equal to \( k_n (t_0 - \tau_{on}) \). (The subscript \( n \) has been appended to the parameters to emphasize that they are generally different for each eigenray path.)

**Model Transfer Function**

**Spectral Characteristics**

The acoustic transfer function of the ocean for a motional source will be the sum of the transfer functions for each of the relevant eigenray paths (Fig. 2). Thus, from (12)

\[ H(i\omega; t) = \sum_{n=1}^{N} H_n(i\omega; t) \]

\[ = \sum_{n=1}^{N} A_n e^{-i\omega [\tau_{on} - (k_n - 1) (t - \tau_{on})]} \]

(14a)

The received signal \( r(t) \) for a pure sinusoidal source signal will be

\[ r(t) = H(i\omega; t) e^{i\omega t} = \sum_{n=1}^{N} A_n e^{i\omega_0 (t - \tau_{on})} \]

(14b)
The above equation states, in effect, that the received signal is the weighted superposition of a number of sinusoids of different frequency and phase shift. In addition, the amplitude and frequency (or phase) of the signal components vary slowly with time as a result of the changing position of the source and the motional and medium fluctuations. The magnitudes of the amplitude variations are determined primarily by the vertical spreading function \( \alpha_n(R; \chi) \), while the magnitudes of the frequency (or phase) variations are determined by the combined fluctuations of the medium and the source motion.

It is convenient to write the received signal as

\[
r(t) = A_m e^{i \mathbf{k} \mathbf{m} \omega (t - \tau_m)} \sum_{n=1}^{N} A_n e^{-i \theta_n(\omega; t)} = A_m e^{i \mathbf{k} \mathbf{m} \omega (t - \tau_m)} H'(\omega; t)
\]

where \( H'(\omega; t) \) is the modified ocean transfer function, \( A_n' = A_n/A_m \), and

\[
\theta_n(\omega; t) = \omega [k_m (\tau_m - t) - (k_n - k_m) (t - \tau_m)] \\
= 2 \pi [k_m (\tau_m - t) f] - \int_{\tau_m}^{t} \Delta \Psi_n dt.
\]

The modified transfer function provides a measure of the received-signal amplitude and phase characteristics relative to the reference sinusoid \( A_m \times \exp (i \mathbf{k} \mathbf{m} \omega (t - \tau_m)) \). The parameters of the reference sinusoid may be chosen to minimize the translation and variation of the time delay and frequency parameters. To accomplish this, the values of \( \tau_m \) and \( k_m \) will lie somewhere between the extremes of the values of \( \tau_{on} \) and \( k_n \) over all \( n \). In the case where one eigenray signal (say \( A_q \)) dominates over the analysis interval, then a suitable choice for \( A_m \), \( \tau_m \), and \( k_m \) would be: \( A_m = A_q, \tau_m = \tau_{on}, \) and \( k_m = k_{on} \), where the bar over the parameter indicates its mean value over the analysis interval. It is not necessary, however, that the reference-signal parameters be specifically related to any of the corresponding eigenray parameters.

The instantaneous frequency \( \Delta \Psi_n \) [in (15b)] is the time derivative of the variable phase term \( (k_n - k_m) (t - \tau_m)f \). This frequency can be written [from (2)] as

\[
\Delta \Psi_n = \left[ \frac{v_0}{c} \cos \Psi_0 \cos \varphi_m - \cos \varphi_m \right] \left( t - \tau_{on} \right) f + k_n^2 \epsilon_n \int_{\tau_{on}}^{t} \Delta \Psi_n dt.
\]

Using the modified transfer function, the amplitude and phase characteristics of the received signal (relative to the reference signal) may be computed as

\[
H'(i \omega; t) = A(\omega; t) e^{-i \theta(\omega; t)}
\]

where the amplitude characteristics are

\[
A(\omega; t) = \left[ \sum_{n=1}^{N} A_n \cos \theta_n \right]^2 + \left[ \sum_{n=1}^{N} A_n \sin \theta_n \right]^2
\]^{1/2}

and the phase characteristics are

\[
\theta(\omega; t) = \tan^{-1} \left( \sum_{n=1}^{N} A_n' \sin \theta_n \right) \sum_{n=1}^{N} A_n' \cos \theta_n
\]

Impulse Response

The impulse response of the ocean for a motional source becomes [from (13)]

\[
h(t; t_0) = \sum_{n=1}^{N} B_n(x_0) e^{-4 \pi f_n^2 (t - t_{on})^2} + (k_n - 1) f_n\]

This relation shows that the impulse response of the ocean is the weighted superposition of Gaussian pulses with various means and standard deviations. (An example of the ocean impulse response is demonstrated in [14, Fig. 4].)

System Response to an Arbitrary Signal

The received output for an arbitrary signal \( u(t) \) can be computed by convolving the source signal with the impulse response. Thus,

\[
r(t) = \int_{-\infty}^{\infty} u(\xi) h(t; \xi; t_0) d\xi
\]

\[
\approx \sum_{n=1}^{N} B_n(x_0) \int_{0}^{\infty} u(\xi) e^{-4 \pi f_n^2 (t - \tau_n)^2} d\xi.
\]

The above relation is not reducible to closed form in the general case. However, if the power spectral density of \( u(t) \) has an upper bound appreciably less than the information bandwidth \( f_{on} \) for all \( n \), the impulse response can be treated as a Dirac delta function with little error. In other words, if the change in \( u(t) \) over temporal increments of \( 1/f_{on} \) (approximate width of the Gaussian pulse) for all \( n \) is very small, the impulse response can be treated as a Dirac delta function with little error. Under these circumstances the system response can be approximated as

\[
r(t) \approx \sum_{n=1}^{N} A_n u(k_n (t - \tau_{on})).
\]

Thus, the received signal is the weighted superposition of a number of source signals both slightly compressed (or expanded) and translated in time. When \( u(t) \) is a sinusoid equal to \( e^{i \omega t} \), (19) is seen to reduce to (14b).

Application of the Model Transfer Function

Application of the model transfer function will permit a rather detailed comprehension of the distortion suffered by a signal after propagating over long ranges in the deep ocean. In addition, the transfer function may be used to study and
analyze the coherence properties of signals received over remote paths. Two examples of the use of the transfer function to predict the remotely received signal characteristics will now be given.

Filter Characteristics

As one might anticipate, the ocean medium behaves as a complex filter for acoustic signals. To demonstrate this, consider the acoustic source as fixed and eigenray parameters as given in Table I. (These parameters were crudely estimated from the impulse response of the ocean over a range of approximately 250 nmi.) The reference signal is chosen as the strongest of the received eigenray signals (viz. \( n = 6 \)). The modified transfer characteristics over the frequency range of 30 to 40 Hz were computed using (17) and are shown in Fig. 6. It will be observed that the amplitude and phase variation can exceed 20 dB and 90 degrees, respectively, for a frequency shift of only a small fraction of a hertz.

To better understand the filter characteristics, Fig. 7 depicts in vector form the amplitude and phase relations of the eigenray signals for a frequency of 33 Hz. The received signal is shown lagging the reference eigenray signal by 24.1 degrees. As the frequency is increased above 33 Hz, the eigenray signal vectors will rotate relative to the reference vector at a rate proportional to the moment arm \( \tau_{on} - \tau_{o6} \). (For a positive moment arm the rotation will be clockwise, while for a negative moment arm the rotation will be counterclockwise.) The amplitude of the resultant received signal will peak when all of the eigenray signal vectors are closely aligned in phase (see Fig. 6). On the other hand, when the resultant vector is small its phase can be expected to lie anywhere over the vector plane. It should be evident that the larger the relative amplitude of the eigenray signal, the greater will be its influence in determining the amplitude and phase of the received signal. Thus, highly attenuated eigenray signals resulting from highly lossy boundary interactions can generally be ignored in practical applications of the transfer function.

Motional Effects

To demonstrate the motional effects, the same eigenray parameters were used for a CW frequency of 33 Hz and a source range-rate of 10 knots. It is assumed that the eigenray amplitude parameters are constant over the relevant analysis interval. The computed transfer characteristics are shown in Fig. 8. It will be noted that a rather wide variation in both amplitude and phase of the received signal can be realized over a matter of a few minutes in the example. The variation may again be deduced by a study of Fig. 7 which shows the initial alignment of the eigenray signal vectors at the onset of the source run. Although the source frequency is constant, the frequency of each eigenray signal will be slightly different, causing the various eigenray signal vectors to rotate at a rate proportional to the Doppler difference-frequency \( \Delta f_n \). (The direction of the rotation will be clockwise for a negative Doppler difference-frequency and counterclockwise for a positive Doppler difference-frequency.) The continuous variation of the eigenray signal phases over time produces the resulting amplitude and phase characteristics shown in Fig. 8.

Although the above transfer characteristics were computed for fixed eigenray parameters, it is evident that the multipath nature of long-range acoustic propagations can significantly distort the signal emanating from a moving source. In practice the eigenray parameters can be expected to vary with time, due to the changing eigenray paths with source range and due to the fluctuations inherent in the medium (including its boundaries) and the source motion. Compounding this with source frequency variations, it is evident that the temporal coherence of acoustic signals received over long distances in the deep ocean can suffer rather serious degradation.

Summary and Conclusions

An acoustic transfer function is available to assess the distortion suffered by a signal emanating from a motional source after propagating over long distances in the ocean. The transfer function is based on the discrete multipath propagation model predicted by geometrical acoustics, and assumes a dispersionless medium. The relevant parameters are the relative ampli-
Fig. 8. Example ocean transfer characteristics for a motion CW source signal ($f = 33$ Hz and $R = 10$ knots). Note: Dashed line indicates phase lies in the second and third quadrants.

titude, the relative propagation time, and the ray angle (at the source) of the significant eigenray signals over the analysis interval. A remotely received signal consists of a weighted sum of modified source-signals, both slightly compressed (or expanded) and translated in time. The compression factor is itself time variable, creating accordion-like ripples along the source-signal time-scale. The variance of the time-scale fluctuations is proportional to the variance of the effective source motional fluctuations, and is dependent on the spectral distribution of these fluctuations. The more the spectral power is concentrated at a frequency equal to the inverse of the analysis interval, the greater will be the variance of the time scale-factor distortion. Although the standard deviation of the time scale-factor fluctuations will generally be only a small fraction of a second in many practical applications, this can seriously degrade the temporal coherence of the received signal, particularly at the higher signal frequencies [7].

Applications of the transfer function to a specific ocean profile demonstrated the nature of the ocean filter characteristics, and the complex nature of the received signal from a CW source in motion. The model transfer function can be a useful tool for signal processing applications in underwater acoustics.

**APPENDIX**

**Solution of $k(t)$**

The solution for the time scale-factor function $k$, given by (4b), is in terms of signal time $x$ at the source. Since the time $t$ that each signal event arrives at the receiver is simply $x + \tau(x)$, the interrelationship between these two variables may be used in (4b) to yield

$$x(t) = k(t - \tau_0)$$

and

$$k = 1 + \frac{1}{t - \tau_0} \int_0^{k(t - \tau_0)} \frac{V \cdot l_x}{c} dx$$

$$= k_0 \left[ 1 + \frac{e}{t - \tau_0} \int_0^{k(t - \tau_0)} \xi(x) dx \right].$$

(A2)

Thus, to determine $k$ in terms of signal time $t$ at the receiver requires the solution of the above integral equation.

To solve the integral equation, let $k$ take the form

$$k = \frac{1}{k_0^\eta - \eta} = k_0 \left[ 1 + \frac{k_0 \eta}{1 - k_0 \eta} \right]$$

(A3)

where $\eta$ is a new variable much smaller than unity. Substituting (A3) in (A2) and reducing gives

$$\eta = \frac{e(1 - k_0 \eta)}{k_0(t - \tau_0)} \left\{ \int_0^{k_0(t - \tau_0)} \xi(x) dx \right\} + \left\{ 1 + (k_0 \eta / 1 - k_0 \eta) \right\} k_0(t - \tau_0) \xi(x) dx \right\}. \quad (A4)

Since $\eta \ll 1$ and $\xi(x)$ is a relatively slowly varying function, the second integral in (A4) may be evaluated using the mean value theorem for integrals. Thus, carrying out the indicated algebra, the solution for $\eta$ may be shown to be

$$\eta = \frac{e(\langle \xi(x) \rangle)k_0(t - \tau_0)}{1 + k_0 e(\langle \xi(x) \rangle)k_0(t - \tau_0) - \frac{e}{k_0(t - \tau_0)} \langle \xi(x) \rangle k_0(t - \tau_0)}$$

(A5)

where

$$\langle \xi(x) \rangle k_0(t - \tau_0) = \frac{1}{k_0(t - \tau_0)} \int_0^{k_0(t - \tau_0)} \xi(x) dx \quad (A6)$$

is a form of running time average of the fluctuating variable $\xi(x)$. Because $e \ll 1$, the solution for $k$ may be closely approximated as

$$k = k_0 \left[ 1 - \frac{e}{t - \tau_0} \int_0^{k_0(t - \tau_0)} \xi(x) dx \right]^{-1}$$

$$= \left[ 1 - \frac{V \cdot l_x}{c} k_0(t - \tau_0) \right]^{-1}.$$

(A7)

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**REFERENCES**


[8] The equation is obtained from [7, equations (3-8) and (3-24)]. This reference contains analyses closely related to the problem discussed in this report and may be used to supplement the results derived herein.

[9] The analyses in this section represents an extension and elaboration of the results presented in [7, Appendix].


Some Novel Windows and a Concise Tutorial Comparison of Window Families

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Abstract—Some novel windows are introduced. A comparison of these and the well-known windows in terms of their frequency domain properties is given. It is concluded that Kaiser, modified Kaiser, Tukey, and three-coefficient window families appear to be the best of the known windows of 6, 12, and 18 dB/oct decay rates.

I. INTRODUCTION

There are two basic methods of power spectra estimation [1]. One is the indirect method in which the autocorrelation of the process is estimated, windowed by a lag window (the Fourier transform of which is called a spectral window), and then Fourier transformed. The other is the direct method in which the power spectrum of the process is estimated as the energy spectrum of the process windowed by a data window (the Fourier transform of which is called a frequency window). In the latter method, in order to reduce variance of the estimate, one may smooth it by convolving with a spectral window. Hence, in the indirect method, a spectral window is unavoidable, while in the direct method, a data window is unavoidable but a spectral window is used to reduce the variance.

There are many windows suggested or derived to optimize some features of a window. Let \( w(t) \) designate a lag or data window and \( W(f) \) be its Fourier transform. Then, the common properties of these proposed windows can be summarized as follows:

1) \( w(t) \) is real, even, and nonnegative.
2) \( |W(f)| \) has a main lobe at the origin and sidelobes at both sides (Fig. 1).
3) If the \( m \)th derivative of \( w(t) \) is impulsive, then the peak of the sidelobes of \( |W(f)| \) decays asymptotically as \( 6m \) dB/oct.
4) No window is the best in all aspects, and one should select one according to the requirements of a particular application.

Bias caused by a window \( w(t) \) is

\[
B = [S(f) * W(f)] - S(f)
\]

if \( w(t) \) is used as a lag window, or

\[
B = [S(f) * |W(f)|^2] - S(f)
\]

if \( w(t) \) is used as a data window, where \( S(f) \) designates the expected value of the unwindowed estimate [1]. Obviously, bias depends on the shape of \( W(f) \) or \( |W(f)|^2 \). However, in general, it is difficult to determine how bias depends on the shape of \( W(f) \), for it changes also with \( S(f) \). Therefore, it is quite reasonable to attribute the undesirable bias effect to the following three sources: resolution degradation due to the main lobe, leakage due to the near sidelobes, and leakage due to the far sidelobes. In the following section, a comparison of the windows in terms of four parameters which reflect the effect of these sources of degradation will be presented.

II. COMPARISON OF WINDOWS

Without loss of generality, let \( w(t) \) be unity at the origin, and time limited to the interval \( |t| < \frac{1}{2} \), i.e.,

\[
w(0) = \int_{-\infty}^{\infty} W(f) df = 1
\]

\[
w(t) = 0 \quad \text{for} \ |t| > \frac{1}{2}.
\]