method. At present, the above algorithm is generalized to include higher order modes for which the block-Toeplitz structure of the impedance matrix remains valid.

Experimental work is presently in progress. Comparison with the numerical results will soon be carried out. The scope of this work may also be broadened to include symmetrical nonuniformly spaced arrays.

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REFERENCES


An Inverse Electromagnetic Algorithm for Aspect-Limited Data

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Abstract—An algorithm which determines the shape of a conducting convex scatterer from the depolarization properties of the (monostatic) scattered signal is shown to be effective when the return signal is from a limited range of aspects. Numerical examples are included.

I. INTRODUCTION

It is a long-standing problem in electromagnetic scattering theory to determine the shape of a body from the scattered field it creates in interacting with sampling radiation. The shape-recovery methods resulting from the many investigations in inverse scattering are well reviewed in [1]. More recently, there has been a growing interest in using the vector nature of the problem as a means of extending the type of information available for reconstruction algorithms [2].

The present discussion derives from this polarization/depolarization approach and further develops a numerical inversion technique to applications involving more limited data sets (and, more realistic data sets) than previously examined. Specifically, we established a method that allows the shape of a conducting convex body to be determined from a sparse data set (obtained as a function of aspect) and applied this technique to sample data sets defined over all aspects [3]. As few radar encounters can be expected to yield such complete information, we shall show how the technique can be applied to similar data defined over a limited range of aspects.

In doing this, we first review the existing theory (Section II) and then explain how aspect-limited data can be accommodated within this theory (Section III). Section IV summarizes the numerical results of applying the developed algorithms to sample data sets.

II. THE VECTOR INVERSE ALGORITHM

The approach we shall take results from the observation that the principal radii of curvature local to the specular points of a perfectly conducting convex body are radar measurables [3]-[6]. Illuminating such a target with an impulse \( H_I = H_o \tilde{u} + H_o \tilde{v} = H_0 (\tau - z/c) \) will effectively isolate the specular points, and it can be shown that the leading edge of the return in the

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far-field approximation obeys [3], [7]

$$H_x = \mathbf{S}\mathbf{H} = \frac{1}{2\pi cd} \left[ \frac{1}{c} \frac{\partial^2 A_x}{\partial t^2} \right],$$

$$+ \frac{1}{2} \frac{\partial A_x}{\partial t} (K_u - K_v) (H_u + H_v),$$

where $H_x$ is the scattered magnetic field, $d$ is the distance from the transmitter/receiver to the target, $A_x$ is the target area projected onto the $u - v$ plane (the so-called silhouette function), $K_u$ and $K_v$ are the curvatures in the principal directions $\hat{u}$ and $\hat{v}$ at the specular point, respectively, and $S$ is the scattering matrix in the $u - v$ frame. This is the Kennaugh-Cosgriff formula [8], [9] with Bennett's first-order correction [7].

Denoting the Fourier transform of the $m - n$ element of the scattering matrix by $s_{mn}(\omega)$, then in a reference coordinate system in which $H = H_1 \hat{t} + H_2 \hat{j}$ is rotated at an angle $\eta$ with respect to the $u - v$ system, it is straightforward to see that [3]-[5]

$$K_u - K_v = \frac{2\omega/c}{\cos 2\eta} \tan \left( \frac{\phi_\eta}{2} \right) \equiv B,$$

(1)

where $\phi_\eta$ is the phase difference between $s_{22}(\omega)$ and $s_{11}(\omega)$ and

$$\cos 2\eta = \left[ 1 + \left( \frac{s_{22}(\omega)}{s_{11}(\omega)} \right)^2 \cos^2 \left( \phi_\eta/2 \right) \right]^{-1/2}.$$

(2)

By virtue of the high-frequency relationship [10]

$$\sigma = \frac{\pi}{K_u K_v},$$

between the differential cross section $\sigma$ and the principal curvatures, we can conclude that $K_u$ and $K_v$ are separately measurable and, in particular,

$$R_u + R_v = \frac{B_0}{\pi} \sqrt{1 + \frac{4\pi}{\sigma B^2}}.$$

(4)

($R_u = 1/K_u$ and $R_v = 1/K_v$ are the principal radii of curvature.)

The significance of this result lies in its relationship to a differential equation developed by Stoker to study the Christoffel-Hurwitz problem of classical differential geometry:

**Can the Surface be Reconstructed from a Knowledge of the Sum of its Principal Radii of Curvature as a Prescribed Function of Direction of Surface Normal?**

Let $(u, v)$ be a parametric representation of a convex surface $S$. Denote the unit normal to $S$ at $r$ by $\hat{n} = (n_1, n_2, n_3)$. Then we can define the spherical image as the mapping induced by

$$h(n_1, n_2, n_3) = r \cdot \hat{n}.$$

The Minkowski support function $h(X_1, X_2, X_3)$ can be defined by requiring that

$$h = r \cdot X = |X|h$$

for any vector $X = (X_1, X_2, X_3)$ parallel to $n$.

In [11], Stoker obtains a differential relationship between $h$ and $\Phi = R_u + R_v$, a relationship which for the present problem reduces to [3]

$$\nabla^2 h = X^2 \nabla^2 h + 2h = -\Phi$$

(5)

(expressing $h$ as a function of the spherical coordinates $(\theta, \phi)$ of $\hat{n}$). Since $\mathbf{r} \cdot \mathbf{X} = d\Phi$, the Cartesian coordinates of $S$ can then be written

$$x = h \sin \theta \cos \phi + \frac{\partial h}{\partial \theta} \cos \phi \cos \theta - \frac{\partial h}{\partial \phi} \sin \phi \sin \phi$$

$$y = h \sin \theta \sin \phi + \frac{\partial h}{\partial \theta} \cos \theta \sin \phi + \frac{\partial h}{\partial \phi} \sin \phi \sin \phi$$

$$z = h \cos \theta - \frac{\partial h}{\partial \theta} \sin \phi,$$

(6)

and so the target shape is recovered with the solution of (5).

In radar applications, $\Phi$ rarely will be determined as a prescribed function of aspect, and, because of this, a numerical solution to (5) is required. To this end, we investigate the functional

$$F = \int \left[ (\nabla h)^2 - 2h \Phi \right] dv - 2 \int h \beta dT$$

$$= \int D \left[ \left( \frac{\partial h}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial h}{\partial \phi} \right)^2 \right. - 2h(h + \Phi) \right] \sin \theta d\theta d\phi - 2 \int h \beta dT,$$

(6)

where $D$ is the intersection of $V$ with a sphere of radius 1 centered on $r = 0$. If $h(\theta, \phi)$ is a solution to (5) over $D$ with boundary $T = \delta V$, then $h(\theta, \phi)$ also makes (6) stationary ($\beta$ denotes the outward directed normal derivative of $h$ along $T$, [12]).

Given a discrete set of data $\{h_i\}$ over a sparse set of aspects, we can divide $D$ into $N$ subdomains (one subdomain for each element of $\{\Phi_i\}$) and integrate a linearization $h_i(\theta, \phi)$ of $h(\theta, \phi)$ over each subdomain. Seeking the values $\{h(\theta_i, \phi_i)\}$ that satisfy $\delta F = 0$, we require

$$\frac{\partial F}{\partial h(\theta_i, \phi_i)} = 0, \quad i = 1, \ldots, N,$$

(7)

and solve the resulting set of linear algebraic equations for $\{h(\theta_i, \phi_i)\}$ by the usual methods.

This is the approach taken in [3] under the assumption that the set $\{\Phi_i\}$ had elements distributed over all aspects so that the boundary integral in (6) was identically zero.

**III. ASPECT-LIMITED DATA SETS**

The necessity of acquiring data over all aspects would severely limit the applicability of this technique to radar inverse scattering. Furthermore, while the boundary integral in (6) promises to resolve this drawback, the boundary conditions are not easily determined along $T$ on the spherical image. (These boundary conditions are determined by the scalar product of the position vector of the surface and the normal $\hat{t}$ to $V$ along $T$, $\beta = \hat{t} \cdot \nabla h(T)$.) Fortunately, for many radar inverse applications, we only require the shape of the scatterer up to a translation, and it is precisely this fact that will effectively allow us to obtain a solution over aspect-limited data sets.

Two support functions which differ by only a linear function will have corresponding surfaces which differ by at most a translation [11]. For the Christoffel-Hurwitz problem then, $\beta$ (which determines $S$ up to an additive constant) need not be specified.
Rather, we only have to force the values of the tangent $\partial r/\partial \xi$ along the boundary.

To see this, consider the difference $\delta$ between two support function solutions to (5). Clearly, $\delta$ is harmonic and homogeneous of degree one. Therefore, its derivatives $\delta_i$ are harmonic and homogeneous of degree zero. From the theory of uniqueness of harmonic functions, $\delta_i$ will be determined up to an additive constant from (5) if $\partial \delta_i/\partial \xi$ is specified along $T$. But this will just determine $\delta$ up to a linear function. (This can be made rigorous, c.f. [13]–[16].)

In the following, we shall let the spherical image serve as the parameter surface for $r(u, v)$ with the coordinate curves $u, v$ taken as lines of curvature on $S$.

Define a new (local) parameter system $(\xi, \alpha)$ on the unit sphere by $u = \xi \cos \alpha$ and $v = \xi \sin \alpha$, and set $\alpha$ to be the angle between the direction normal to $T$ and the $u$ parameter curve. Then

$$r_\xi \equiv \frac{\partial r}{\partial \xi} = r_u \cos \alpha + r_v \sin \alpha,$$

By virtue of our choice of parameter surface, the $u - v$ coordinate curves are the unique curves having their tangents parallel to the tangents of their own spherical images. Using Rodrigues’ formula, we can write

$$r_\xi = -R_u \hat{u} \cos \alpha - R_v \hat{v} \sin \alpha,$$

expressing $r_\xi$ as a function of the data already obtained through (1)-(3). Since $\hat{u}$ and $\hat{v}$ are unit vectors in the plane tangent to the unit sphere along $T$ with orientation defined by (2), we can completely specify the required boundary conditions in terms of radar measurables. Consequently, we may now consider the modified problem of determining a function $H'$ that makes

$$\int_V \left[ (\nabla H')^2 - 2H' \right] dV$$

stationary and satisfies

$$\hat{\xi} \cdot \nabla (\nabla H') = -R_u \hat{u} \cos \alpha - R_v \hat{v} \sin \alpha$$

along $T$ and ignores the fact that such a solution will differ from $H$ by some linear function.

IV. NUMERICAL RESULTS

For convenience, all formulas will be developed in spherical coordinates. Dividing the domain of $h(\theta, \phi)$ into $N$ subdomains $D_1$ (with boundaries defined by constant $\theta$ and $\phi$ curves), we approximate over each $D_i$

$$h = h_i(\theta, \phi) = h(\theta_i, \phi_i) + \frac{h(\theta_i + \Delta \theta_i, \phi_i) - h(\theta_i, \phi_i)}{\Delta \theta_i} (\phi - \phi_i) \sin \theta$$

and

$$h(\theta_i + \Delta \theta_i, \phi_i) - h(\theta_i, \phi_i) = \frac{\Delta \phi_i}{\Delta \theta_i} (\theta - \theta_i),$$

where $\Delta \theta_i$ and $\Delta \phi_i$ are the differences between the $\theta$ curves and the $\phi$ curves bounding $D_i$, respectively. Assuming $\Phi = \Phi_i$, a constant, on each $D_i$, the functional of interest becomes

$$F = \sum_{i=1}^{N} \int_{D_i} \left[ \left( \frac{\partial h_i}{\partial \theta} \right)^2 + \frac{1}{\sin^2 \theta} \left( \frac{\partial h_i}{\partial \phi} \right)^2 - 2h_i(h_i + \Phi_i) \right] \sin \theta d \theta.$$

The boundary conditions are implemented in the usual way [12] by restricting the $h_i$ of the “boundary region” nodes to those values required by a finite-difference implementation of (8).

The set of equations (7) readily lends itself to numerical solution [18]. The Fortran code developed was tested on simulated data sets containing the appropriate aspect-limited curvature information for convex sample targets.

For an ellipsoid, this information takes the form [6]

$$R_u + R_v = \frac{-[a^2(b^2 + c^2) \sin^2 \theta \cos^2 \phi + b^2(a^2 + c^2) \sin^2 \theta \sin^2 \phi + c^2(a^2 + b^2) \cos^2 \theta]}{[a^2 \sin^2 \theta \cos^2 \phi + b^2 \sin^2 \theta \sin^2 \phi + c^2 \cos^2 \theta]^{3/2}}$$

for convex sample targets.

All calculations were performed in single precision on an HP-1000/F minicomputer, and a detailed analysis of discretation and round-off errors in the implementation of (9) is beyond the scope of the present discussion (see, for example, [12]). However, an elementary first-order error analysis shows that under the assumptions for which (4) is expected to remain valid [3], the percentage error in $(R_u + R_v)$ is the same as that in $\phi_d$, the measured diagonal phase difference. We are then free to examine the question of robustness against noise defects by perturbing our sample data sets, and this type of (cursory) analysis has shown that the algorithm remains stable under the inclusion of Gaussian (mean zero) noise, and effective if the noise level is less than about 20 percent.
CONCLUSION

The foregoing analysis and examples have demonstrated that the variational solution of the Stoke's differential equation approach to inverse scattering is not limited to full-aspect data. This means that data more representative of the kind obtainable in radar encounters can be used to recapture the shape of convex portions of the scattering body. Numerical testing of the developed algorithms has borne this out. Still unanswered, however, are the questions of how accurately these data can be obtained from actual radar systems and how best to obtain the data. To the author's knowledge, the relation (1) has not been completely verified.

REFERENCES


Correction to "UAT Analysis of E-Plane Near and Far-Field Patterns of Electromagnetic Horn Antennas"

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In (4) of the above paper, \( \exp (jkr_1) \) should be replaced by \( \exp (jkr_1') \).

On page 819, column 1, tenth line from the bottom, "\( \rho \)" should be replaced by "\( \theta \)."

On page 819, second line before the acknowledgment, [8] should be replaced by [6].

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