Electromagnetic Theorems for Complex Anisotropic Media

CLIFFORD M. KROWNE, SENIOR MEMBER, IEEE

Abstract—Complex anisotropic media can generally be described by a 6×6 macroscopic constitutive tensor $\mathbf{C}$. Using $\mathbf{C}$ properties, a reaction integral formula is derived from which an anisotropic reaction theorem (modified reciprocity theorem) is developed. Reduction of the $\mathbf{C}$ medium into a reciprocal medium is discussed including tensor symmetry attributes and limiting cases. The anisotropic reaction theorem is utilized to derive a zero reaction theorem, and then treated in relation to the moment method. Mutual and self-impedance elements of a network are also derived in terms of reaction integrals, symmetry covered using the anisotropic reaction theorem, and impedance elements related to moment calculations. Use of spectral domain analysis is also covered.

I. INTRODUCTION

THE USE of radiating patches, resonators, and transmission lines compatible with integrated circuit technology using new materials today highlights the need to understand how these materials affect conventional methods used to calculate their properties. Gyroelectric, gyromagnetic, and optically active materials, to name a few, may all be used in hybrid and monolithic integrated circuits. They can be nonreciprocal materials, and may involve the design of distributed active devices providing signal gain or oscillation behavior, for example. The general description of a nonmoving anisotropic medium, which can be lossy, may be expressed in terms of a 6×6 macroscopic tensor $\mathbf{C}$. Such 6×6 tensors have been described in optics [1] and electromagnetics [2]. Following the submatrix notation in the optics work, it has been shown that the Green’s function for a complex anisotropic planar layered structure can be conveniently found in the spectral domain [3]. This spectral Green’s function may readily be applied to finding radiator, resonator, or transmission line properties. However, the determination of these properties is based on a number of theorems including the reciprocity theorem, zero reaction theorems, and mutual impedance symmetry theorems. All of these theorems are modified as to their conditions and statements when going from isotropic media to specialized anisotropic media. Furthermore, when considering the most general anisotropic medium, say in the form of a $\mathbf{C}$ tensor, the theorems are again modified.

It is the purpose of this paper to review and extend the theorems developed for or used in 1) isotropic media applications [4]-[8], and 2) anisotropic media applications [2], [8], [9]. Section II derives a reaction integral formula, and from this formula an anisotropic reaction theorem (or modified reciprocity theorem). Conditions which lead to the anisotropic reaction theorem are delineated. Tensor symmetry properties of $\mathbf{C}$ and specialized $\mathbf{C}$ cases which impact on whether or not a medium can be identified as reciprocal (versus nonreciprocal) are presented. A zero reaction theorem is obtained from the anisotropic reaction theorem in Section III. A moment expression method of the zero reaction theorem is provided. Section IV discusses mutual and self-impedance network elements using reaction integrals, symmetry employing the anisotropic reaction theorem, and the use of the moment method. The advantage of doing calculations in the spectral domain is pointed out in Sections III and IV.

II. DERIVATION OF ANISOTROPIC REACTION THEOREM

In this section the analog to the reciprocity theorem in isotropic space is derived with a particular representation in mind for the constitutive relationships. Define the six-element column vectors $\mathbf{A}_R$ and $\mathbf{A}_L$ (superscript $T$ denotes transpose) as

$$ \mathbf{A}_R = [D_x D_y D_z B_x B_y B_z]^T $$

(1a)

$$ \mathbf{A}_L = [E_x E_y E_z H_x H_y H_z]^T $$

(1b)

where $D_i$, $B_i$, $E_i$, and $H_i$ are, respectively, the electric displacement, magnetic displacement, electric, and magnetic field components. Vectors $\mathbf{A}_R$ and $\mathbf{A}_L$ can be related by a 6×6 constitutive tensor $\mathbf{C}$,

$$ \mathbf{A}_R = \hat{\mathbf{C}}\mathbf{A}_L, $$

(2)

specifying $\hat{\mathbf{C}}$ to be

$$ \hat{\mathbf{C}} = \begin{bmatrix} \hat{\epsilon} & \hat{\mu} \\ \hat{\mu} & \hat{\epsilon} \end{bmatrix}. $$

(3)

$\hat{\mathbf{C}}$ has special importance if constructed in the fashion of (3) because it can be used in matrix methods of finding radiation, resonant, or propagation properties of planar structures consisting of complex anisotropic layered media [3]. $\mathbf{C}$ is composed of four 3×3 tensors, the permittivity tensor $\hat{\epsilon}$, the permeability tensor $\hat{\mu}$, and the optical activity tensors $\hat{\mu}$ and $\hat{\mu}'$.

Maxwell’s equations assuming field proportionality to $\exp(j\omega t)$ are

$$ \nabla \times \mathbf{E} = j\omega \mathbf{B} + \mathbf{M}, $$

(4a)

$$ \nabla \times \mathbf{H} = j\omega \mathbf{D} + \mathbf{J}, $$

(4b)

with $\mathbf{M}$ and $\mathbf{J}$ being, respectively, volume magnetic and electric currents. Consider two different sets of source currents ($\mathbf{I}_a$, $\mathbf{M}_a$) and ($\mathbf{J}_b$, $\mathbf{M}_b$) which produce, respectively the fields ($\mathbf{E}_a$, $\mathbf{H}_a$) and ($\mathbf{E}_b$, $\mathbf{H}_b$) in the inhomogeneous anisotropic original medium. The original medium has the following interpretation. It constitutes all space including interfaces, different obstacle materials, conductor walls, and their specific physical disposition in relation to one another. Using, respectively, the last three rows or the first three of (2) in (4a) or (4b), field source case $a$ or $b$ yields

$$ \nabla \times \mathbf{E}_{a,b} = j\omega(\hat{\mu}'\mathbf{E}_{a,b} + \hat{\mu}\mathbf{H}_{a,b}) + \mathbf{M}_{a,b}, $$

(5a)

$$ \nabla \times \mathbf{H}_{a,b} = j\omega(\hat{\epsilon}\mathbf{E}_{a,b} + \hat{\mu}\mathbf{H}_{a,b}) + \mathbf{J}_{a,b}. $$

(5b)

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The author is with the Electronics Technology Division, Naval Research Laboratory, 4555 Overlook Avenue, S. W. Washington, D. C. 20375.

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Now consider a new problem to be solved whereby the original medium has its constituent materials replaced by different materials. Interfaces, boundaries, and walls are left untouched, however. This new problem, hereafter referred to as the altered problem, generates different fields from the original problem for the same sources. Statements equivalent to (5) for the altered problem are

\[ \nabla \times \tilde{\mathbf{E}}_{a,b} = 0, \]

where tildes over the field components and the 3 × 3 tensors denote the altered case.

Form two equations from (5) and (6) by multiplying (5a), case b by \( \tilde{\mathbf{H}}_a \), the first equation, and multiplying (6b), case a by \( \tilde{\mathbf{E}}_b \), the second equation. Add the two resulting equations to obtain

\[
\nabla \cdot (\tilde{\mathbf{H}}_a \times \mathbf{E}_b) = \mathbf{E}_b \cdot \nabla \times \mathbf{H}_a - \mathbf{H}_a \cdot \nabla \mathbf{E}_b = j_0 [\mathbf{E}_b \cdot \tilde{\mathbf{E}}_a + \mathbf{E}_b \cdot \tilde{\mathbf{H}}_a + \mathbf{H}_a \cdot \tilde{\mathbf{E}}_b + \mathbf{H}_a \cdot \tilde{\mathbf{H}}_b + \tilde{\mathbf{E}}_a \cdot \mathbf{J}_a + \tilde{\mathbf{H}}_a \cdot \mathbf{M}_a],
\]

(7)

In a fashion similar to the creation of (7), multiplying (5b), case b by \( \tilde{\mathbf{E}}_b \) to form one equation and multiplying (6a), case a by \( \mathbf{H}_b \) to form the other equation and adding equations,

\[
\nabla \cdot (\tilde{\mathbf{H}}_a \times \tilde{\mathbf{E}}_a) = \mathbf{E}_a \cdot \nabla \times \mathbf{H}_b - \mathbf{H}_b \cdot \nabla \mathbf{E}_a = j_0 [\mathbf{E}_a \cdot \tilde{\mathbf{E}}_b + \mathbf{E}_a \cdot \tilde{\mathbf{H}}_b + \mathbf{H}_b \cdot \tilde{\mathbf{E}}_a + \mathbf{H}_b \cdot \tilde{\mathbf{H}}_a + \tilde{\mathbf{E}}_b \cdot \mathbf{J}_b + \tilde{\mathbf{H}}_b \cdot \mathbf{M}_b],
\]

(8)

Subtracting (8) from (7) enables the construction of a single equation leading to an anisotropic reaction relationship between fields and current sources.

\[
\nabla \cdot (\tilde{\mathbf{H}}_a \times \mathbf{E}_b - \mathbf{H}_b \times \tilde{\mathbf{E}}_a) = j_0 [\mathbf{E}_b \cdot \tilde{\mathbf{E}}_a - \mathbf{E}_a \cdot \tilde{\mathbf{E}}_b] + j_0 [\mathbf{H}_b \cdot \tilde{\mathbf{H}}_a - \mathbf{H}_a \cdot \tilde{\mathbf{H}}_b + \tilde{\mathbf{E}}_a \cdot \mathbf{J}_b + \mathbf{J}_a \cdot \mathbf{M}_b + \tilde{\mathbf{E}}_b \cdot \mathbf{J}_a + \mathbf{J}_a \cdot \mathbf{M}_b].
\]

(9)

To find a relationship from (9) which only relates field-current vector dot products requires that the left-hand-side (LHS) of (9) be reduced to zero and that the first four square bracketed terms on the right-hand-side (RHS) of (9) be eliminated. All square bracketed terms on the RHS have the form

\[
P = \mathbf{A}^T \cdot \tilde{\mathbf{E}}_b + \tilde{\mathbf{B}}^T \cdot \tilde{\mathbf{H}}_b + \tilde{\mathbf{B}}^T \cdot \tilde{\mathbf{E}}_a + \tilde{\mathbf{H}}_a \cdot \mathbf{A}.
\]

(10)

\[
P = \mathbf{A}^T \cdot \tilde{\mathbf{E}}_b + \tilde{\mathbf{B}}^T \cdot \tilde{\mathbf{H}}_b + \mathbf{A}^T \cdot \mathbf{E}_b + \mathbf{B}^T \cdot \mathbf{B}_b.
\]

(10)

\[
P = \mathbf{A}^T \cdot (\tilde{\mathbf{H}}_a + \mathbf{H}_b) \cdot \tilde{\mathbf{E}}_a.
\]

(10)

\[
P = \mathbf{A}^T \cdot (\mathbf{H}_a + \mathbf{H}_b) \cdot \tilde{\mathbf{E}}_a.
\]

(10)

The LHS of (14), third statement, serves to define the reaction integral to be utilized here. \( \langle b, a \rangle \) is the reaction of field \( b \) due to source \( a \), the first integral on the RHS of (14). Fields and sources in \( \langle b, a \rangle \) all take place in the original medium. Unlike \( \langle b, a \rangle \), reaction integral \( \langle a, b \rangle \), although defined formally in a manner similar to \( \langle b, a \rangle \), alters the medium. That is, \( \langle a, b \rangle \) denotes the reaction of field \( a \) (due to source \( b \)) in the altered medium on source \( b \).

The LHS of (14) will be zero if the integrand is zero everywhere on \( S \). This occurs under several widely varying conditions:

1) \( \tilde{\mathbf{H}}_a \times \mathbf{E}_b = \mathbf{H}_b \times \tilde{\mathbf{E}}_a \); all fields generally nonzero.

2) \( \tilde{\mathbf{H}}_a \mathbf{E}_b \sin (\theta_{a,b}) = \mathbf{H}_b \tilde{\mathbf{E}}_a \sin (\theta_{a,b}); \) subscripts \( t \) indicate tangential to surface \( S \), all tangential components nonzero, \( \theta_{a,b} \) angle between fields.

3) \( \tilde{\mathbf{H}}_a = \mathbf{H}_b = 0 \) on \( S_h \), \( \tilde{\mathbf{E}}_a = \mathbf{E}_b = 0 \) on \( S_e \); where \( S_h \) = magnetic wall surface (not necessarily connected) and \( S_e \) = electric wall surface (not necessarily connected).

\( \mathbf{S}_e + \mathbf{S}_h = \mathbf{S}. \)
4) $\mathbf{H}_a = \mathbf{H}_b = 0$ on $S_h$, $\mathbf{E}_b = \mathbf{E}_a = 0$ on $S_e$.

Conditions 1–4 are listed in order of increasing number of zero field components. Satisfaction of condition 4 also implies conditions 1–3, and condition 3 implies condition 2. Condition 3 has its particular form since on an electric wall $E_1 = 0$ and $H_1 = J_z$ due to continuity across a surface interface, eliminating the possibility of simultaneously having null tangential electric and magnetic fields. For a magnetic wall, condition 3 agrees with $H_1 = 0$ and $E_1 = M_x, J_z$ and $M_y$ are, respectively, surface electric and magnetic currents. There is one exception, or specialization to conditions 3 and 4. This is the circumstance where all the field components limit to zero as infinite distance is approached from the finite sources, the surface $S$ being an expanding surface moving away from the sources. Regardless of which condition is upheld, but assuming that at least one condition is met, the LHS integral in (14) becomes

$$\langle b, a \rangle = \langle \tilde{a}, b \rangle. \quad (15)$$

Statement (15) is not an assertion of reciprocity cast in reaction form. That conclusion follows since the original medium considered is postulated to be arbitrarily anisotropic [see (3)]. Equation (15) equates two reaction integrals, one in the original medium, the other in the altered medium, and will be called the anisotropic reaction statement or theorem. Statement (15) considered is postulated to be arbitrarily anisotropic [see (3)].

Equation (15) equates two reaction integrals, one in the original medium. Thus if the altered (or complementary) medium becomes identical to the original medium. Thus if

$$\langle b, a \rangle = \langle a, b \rangle. \quad (17)$$

Reciprocity arises by (16) if

$$\tilde{e}^T = \tilde{e}, \quad (18a)$$

Reciprocity arises by (16) if

$$\tilde{\mu}^T = \tilde{\mu}, \quad (18b)$$

Reciprocity arises by (16) if

$$\tilde{\beta}^T = -\tilde{\beta}. \quad (18c)$$

Reciprocity arises by (16) if

\[ \tilde{\epsilon}_T = \tilde{\epsilon}_d \quad (19) \]

Reciprocity arises by (16) if

\[ \tilde{\epsilon}_T = \tilde{\epsilon}_d \quad (20) \]

Reciprocity arises by (16) if

$$\tilde{\epsilon}_T = \tilde{\epsilon}_d. \quad (20)$$

Expression (20) has only the null tensor solution, $\tilde{\epsilon}_d = 0$. Thus a mixed tensor $\tilde{\epsilon}$ implies nonreciprocity because $\tilde{\epsilon}_d = 0$ is violated. For a biaxial dielectric medium, on the other hand, $\tilde{\epsilon}_d = \tilde{\epsilon}_{22}, \tilde{\epsilon}_{ba} = 0$, and reciprocity (17) holds (considering $\tilde{\mu} = \mu T, \tilde{\beta} = \beta T = 0$. $\tilde{\epsilon}$ for the biaxial case can be diagonalized by rotation of the axes to the principal axis coordinate system. Consider $\tilde{\epsilon}_E$ for a gyroelectric medium, possibly created by applying a dc magnetic field $B_0$ to an ionized gas containing charged ions and electrons or to a solid state gas composed of electrons and holes.

\[ \tilde{\epsilon}_E = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ -\epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \]

\[ \tilde{\mu}_E = \begin{bmatrix} 0 & 0 & 0 \\ 0 & \epsilon_{12} & 0 \\ -\epsilon_{12} & 0 & 0 \end{bmatrix} \quad (21) \]

Integral (14) becomes

$$\tilde{\epsilon}_E = \begin{bmatrix} \epsilon_{11} & \epsilon_{12} & 0 \\ -\epsilon_{12} & \epsilon_{22} & 0 \\ 0 & 0 & \epsilon_{33} \end{bmatrix} \]

Thus a mixed tensor $\tilde{\epsilon}$ can be decomposed into a symmetric part $\tilde{\epsilon}_S$ and an antisymmetric part $\tilde{\epsilon}_A$.

Equation (18a) can be examined by expressing $\tilde{\epsilon}$ as the sum of a symmetric part $\tilde{\epsilon}_S$ and an antisymmetric part $\tilde{\epsilon}_A$.

Equation (18a) employing (19) requires

$$\tilde{\epsilon}_S = \tilde{\epsilon}_d. \quad (20)$$

III. ZERO REACTION THEOREM

Imagine an anisotropic medium, in which the fields are desired, to be interspersed with obstacles or scatterers whose totality of unconnected surface $S$ encloses anisotropic media in which the fields are not necessarily wanted. Fig. 1(a) shows this field problem where all obstacles comprising volume $\Gamma_s$ enclosed by $S$ are represented by a single surface and all sources are represented by $J_j$ and $M_j$ found throughout the volume $\Gamma_j$. The entire medium making up the problem in Fig. 1(a) is characterized by the tensor $\tilde{C}$ which can vary spatially throughout the medium. $\tilde{C}$ is precisely the same constitutive tensor introduced in Section II. Use of the equivalence principle enables the fields in $\Gamma_s$ (inside of $S$) to be replaced by (0, 0) provided the source current $J_s$ and $M_s$ flow on $S$ as seen in Fig. 1(b). Surface currents are calculable from Fig. 1(a) fields on $S$ by $J_s = n \times H_s$ and $M_s = E_{s1} \times n$ where $E_{s1}$ and $H_{s1}$ are on $S$ and $n$ is the unit normal vector directed outward on $S$. Fig. 1(b) will be called the original problem and $\tilde{C}$ the original medium. Fields $\mathbf{E}, \mathbf{H}$ outside of $S$ in Fig. 1(b) are identical to those in Fig. 1(a).

Fields $\mathbf{E}, \mathbf{H}$ in Fig. 1(b) are expressible as a superposition of the scattered fields $\mathbf{E}_s, \mathbf{H}_s$ generated by $J_s, M_s$ (Fig. 2(a)) and the impressed fields $\mathbf{E}_i, \mathbf{H}_i$ caused by the impressed sources $J_i, M_i$ (Fig. 2(b)). $J_j, M_j$ may be thought of as induced by $J_j, M_j$. Thus $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_i$ and $\mathbf{H} = \mathbf{H}_s + \mathbf{H}_i$. Fig. 3 shows test sources $J_T, M_T$ placed within volume $\Gamma_s$, interior to $S$, producing fields $\mathbf{E}_T, \mathbf{H}_T$ interior and exterior to $S$ in $\tilde{C}$.

Transforming medium $\tilde{C}$ in the original problem (Fig. 1(a)) to the altered complementary medium $\tilde{C}$ according to (12) changes the fields from $\mathbf{E}, \mathbf{H}$ to $\mathbf{E}_a, \mathbf{H}_a$ as seen in Fig. 4(a). Particularly noticeable among the changes the transition to $\tilde{C}$ causes is the conversion of the null field within $S$ to $\tilde{E}, \tilde{H}$. Although equivalent sources $J_a, M_a$ in the presence of impressed sources $J_i, M_i$ are of the correct spatial distribution, direction, and magnitude on $S$ to produce exactly a net null field in $\Gamma_a$ in medium $\tilde{C}$, there is no reason to expect this to persist in $\tilde{C}$ because the equivalence principle was not applied in the complementary medium. Fields in the complementary medium $\mathbf{E}_a, \mathbf{H}_a$ are decomposable into scattered and impressed parts such
that $\mathbf{E} = \mathbf{E}_s + \mathbf{E}_i$ and $\mathbf{H} = \mathbf{H}_s + \mathbf{H}_i$. Fig. 4(b) illustrates the origination of the complementary scattered fields due to sources $J_s, M_s$ in $\mathbf{C}$. One sees that the existent field inside of $S$ differs in form from that found for medium $\mathbf{C}$ (Fig. 2(b)). This is due to the existence of the total field within $S$ for medium $\mathbf{C}$. Fig. 4(c) illustrates the creation of the complementary impressed fields due to sources $J_i, M_i$ in $\mathbf{C}$.

Let the anisotropic reaction theorem (15) here be used to develop a useful zero reaction theorem. Initially the reactions will be stated under the specialization to conditions 3 and 4, which require the fields to limit to zero at infinite distance from all current sources, covered before in Section II. Thus the anisotropic reaction theorem for the test sources and fields, and scattered sources and field, stated by replacing $b$ with $T$ and $a$ with $s$ is

$$\langle T, s \rangle = \langle \mathbf{S}, T \rangle. \quad (22)$$

An anisotropic reaction theorem for the test sources and fields, and impressed sources and fields, is stated as

$$\langle T, i \rangle = \langle \mathbf{I}, T \rangle \quad (23)$$

by substituting $i$ for $s$ in (22). Adding (22) and (23) yields

$$\langle T, s \rangle + \langle T, i \rangle = \langle \mathbf{S}, T \rangle + \langle \mathbf{I}, T \rangle. \quad (24)$$
Enlisting the fact that $J_s$, $M_s$ exist only on $S$, and that $J_T$, $M_T$ exist only in $\Gamma_s$, and that $J_i$, $M_i$ exist only within $\Gamma_i$, the LHS of (24) becomes

$$
L = \iint_S (E_T \cdot J_s - H_T \cdot M_s) \, da + \iint_{\Gamma_i} (E_T \cdot J_i - H_T \cdot M_i) \, dv
$$

$$
- H_T \cdot M_s \) \, dv.
$$

(25)

The RHS of (24) similarly reduces to

$$
R = \iiint_{V_s} \left[ (\vec{E} + \vec{H}_i) \cdot J_T - (\vec{H} + \vec{H}_s) \cdot M_T \right] \, dv
$$

$$
= \iiint_{V_s} (\vec{E} \cdot J_T - \vec{H}_i \cdot M_T) \, dv.
$$

(26)

Sum of the reactions $L$ in the original medium $C$ can only equal zero if $R = 0$. Volume integral (26) is zero for arbitrarily $J_T$, $M_T$ if the integrand is zero everywhere within $V_s$. Satisfaction of this requirement occurs for $\vec{E} = 0$ and $\vec{H} = 0$ within $\Gamma_s$ for the $C$ medium. However, Fig. 4(a) already demonstrates that the total field solution within $\Gamma_s$ for the altered medium is generally nonzero. Consequently, simultaneous satisfaction of $E = 0$, $H = 0$ for $C$ and $\vec{E} = 0$, $\vec{H} = 0$ for $C$ within $\Gamma_s$ cannot occur, except for special cases. One such special case is when all obstacles are perfect conductors. The implication for the surface integral (25) is that $M_s = 0$ due to the electric wall present at $S$. From (25) the zero-reaction theorem can now be written for an anisotropic medium as

$$
\iiint_S E_T \cdot J_s \, da + \iint_{\Gamma_i} (E_T \cdot J_i - H_T \cdot M_i) \, dv = 0
$$

(27)

where perfectly conducting obstacles are assumed. Equation (27) takes the same form as found for an isotropic medium.

Equation (27) can be used to solve for the fields and current $J_s$ in the regular way by casting (27) in a moment form. Expand the unknown current distribution $J_i$ in a set of $N_g$ basis functions so that

$$
J_s = \sum_{g=1}^{M} \sum_{p=1}^{N_g} \alpha_{pg} J_{pg}
$$

(28)

where the inner sum constructs the current distribution $J_s$ on the $g$th obstacle. $J_s$ is such that it is zero except on the $g$th obstacle surface $S_g$. There are $M$ obstacles. Identifying the test fields by the same double indexing scheme as currents in (28), enables $M_Q$ linear equations to be produced by placing (28) into (27):

$$
\sum_{g=1}^{M} \sum_{p=1}^{N_g} \alpha_{pg} \iint_{S_g} E_{rq} \cdot J_{pg} \, da + \iint_{\Gamma_i} (E_{rq} \cdot J_i - H_{rq} \cdot M_i) \, dv = 0; \quad r = 1, 2, \ldots, Q_q, \quad q = 1, 2, \ldots, M.
$$

(29)

$E_{rq}$ and $H_{rq}$ are generated by the $J^T_{pq}$ test current on the $q$th obstacle. There are a total of $Q_q$ test currents on the $q$th obstacle. Requiring $Q_q = N_g$ makes (29) a simultaneous set of $MN_g$ equations in the same number of unknowns $\alpha_{pg}$. Enforcing $J_{pq} = J^T_{pq}$, $p = r$ and $g = q$, causes the test and scattered current modes to be the same (the Galerkin process). Contraction of $pq$ and $rq$ indices in (29), $pq \rightarrow n$, $rq \rightarrow m$, allows representation in the more familiar matrix form

$$
K_{\alpha} = U
$$

(30)

$$
K_{mn} = \iint_{S_n} E_m \cdot J_n \, da
$$

(31a)

$$
U_m = \iint_{V_i} (E_m \cdot J_i - H_m \cdot M_i) \, dv.
$$

(31b)

Elements of scattered current vector coefficients $\alpha_{q}$ have units of amperes/m where functions $J_q$ are unitless. $K$ and $U$ have, respectively, units of volts-m and watts. Equation (30) can be solved in conjunction with a spectral domain technique by the following procedure. Convert the problem initially from direct space coordinates $x$, $y$, $z$ to reciprocal space coordinates $k_x$, $k_y$, $k_z$ using an appropriate transforming integral operator suitable to the geometry and other requirements of the problem. For example, the transform could be a Hankel or Fourier transform. In transformed space, the test current function response $F(E_m)$ to the application of $F(J_m)$ can be found by determining the spectral Green's function $G_F$. Here $F$ denotes the transforming operation. Spectral electric field is then readily expressed as the simple product $F(E_m) = G_F F(J_m)$. Once $F(E_m)$, $m = 1, 2, \ldots, MN_g$, are found, elements of $K$ and $U$ become available when test electric fields are converted back into direct space and inserted into (31). This field conversion requires the numerical evaluation of $F^{-1}(G_F F(J_m))$.

IV. MUTUAL AND SELF-IMPEDANCE OF A NETWORK

Reference to (14) provides the reaction definition

$$
\langle b, a \rangle = \iint_{V_i} (E_b \cdot J_a - H_b \cdot M_a) \, dv.
$$

(32)

Placing $N$ electrical ports in the medium $C$ constitutes the network. Each port is excited by the current $I_i$, $i = 1, 2, \ldots, N$, causing a voltage response $V_j$ at the $j$th port. The network may be a $N$-port waveguide or transmission line feed network, a $N$-port excited antenna or resonator patch, or a $N$-port excited antenna array, to name a few possibilities. Considering the $i$th port current source to be $I_i \delta [x - x_i(z)] \delta [y - y_i(z)]$, where $x(\cdot)$ is the unit vector giving the current direction flowing between the terminals of the port, (32) says

$$
\langle f, i \rangle = I_i \int_{t_{1i}}^{t_{2i}} E_i \cdot s \, dt
$$

$$
= -I_i V_i(f).
$$

(33)

Equation (33) was derived using $M_s = 0$, $b \rightarrow j$ and $a \rightarrow i$. Line integration in (33) goes from the bottom to top terminals of the $i$th port, leading to the negative of the definition of the potential between terminals $t_{1i}$ and $t_{2i}$, $V_i(f)$. $V_i(f)$ is the voltage at port $i$ due to applied current $I_i$ at port $j$. 
The total field at port \(i\) is

\[ V_i = z_{ii} I_i \]

when all ports except \(j\) are open-circuited. Solving (35) for \(z_{ij}\) and eliminating \(V_i\) through (33), one obtains

\[ z_{ij} = \frac{\langle j, j \rangle}{I_j I_j} \]

Equation (38) states that the complementary network impedance matrix \(\tilde{Z}\) and the original network impedance matrix \(Z\) are transposes, i.e., \(\tilde{Z} = Z^T\).

Expression (36) used to evaluate \(z_{ij}\) can be found using defined integrals in (31). If we assume for discussion sake that each \(i\)th port is the \(i\)th obstacle, application of current source \(I_j\) to port \(j\) produces a surface current \(J_{sf}\) on the \(j\)th obstacle. \(J_{sf}\) is equal to the sum over its basis set,

\[ J_{sf} = \sum_{n=1}^{N_j} a_n(j) J_n(j) \]

Each expansion mode \(J_n(j)\) acts as a source radiating in the anisotropic \(\tilde{C}\) medium, but without the other conducting obstacles present. \(J_n(j)\) causes \(E_{sf}(n)\) which is found in the spectral domain by the product of the spectral Green's function \(G_P\) and \(F(J_n(j))\). The total field at port \(i\) is

\[ E_i = \sum_{n=1}^{N_j} a_n(j) E_{sf}(n) \]

due to the linearity of medium \(\tilde{C}\). Combining (36) and (40) yields

\[ z_{ij} = -\frac{1}{I_j I_j} \int \int \int_{V_i} E_j \cdot J_i \, dv \]

\[ = -\frac{1}{I_j I_j} \sum_{n=1}^{N_j} a_n(j) \int \int \int_{V_i} E_{sf}(n) \cdot J_i \, dv \]

\[ = -\frac{1}{I_j I_j} \sum_{n=1}^{N_j} a_n(j) U_n(j), \]

where \(U_n(j)\) is given by (31b) with \(M_i = 0\). \(U_n(j)\) is the integral caused by the \(I_j\) port current at port \(j\), due to the \(n\)th basis current function. \(a_n(j)\) are calculated using (30), retaining only those basis function coefficients on the \(j\)th obstacle, setting all others to zero. One can verify that this prescription is correct by examining (34) for \(z_{12} + z_{12} = V_1 I_2\). When all ports except \(j\) are open-circuited, solving (35) for \(z_{ij}\) and eliminating \(V_i\) through (33), one obtains

\[ z_{ij} = \frac{\langle j, i \rangle}{I_j I_j} \]

Medium \(\tilde{C}\), the complementary medium, is related to the original anisotropic medium \(\tilde{C}\) by \((12)\). Mutual impedances in the \(\tilde{C}\) medium can be correlated with those in \(\tilde{C}\). Using the same arguments to arrive at (36) in \(\tilde{C}\), in \(\tilde{C}\)

\[ z_{ij} = \frac{\langle j, i \rangle}{I_j I_j} \]

If one of the surface conditions necessary for (15) to hold is not applicable, then the anisotropic reaction theorem can be utilized to express the complementary mutual impedance as

\[ z_{ij} = \frac{\langle i, j \rangle}{I_j I_j} = z_{ji}. \]

V. CONCLUSION

This paper has reviewed and extended a number of electromagnetic theorems for complex anisotropic media in terms of a 6 \(\times\) 6 macroscopic constitutive tensor. It has been found that a zero reaction theorem for an anisotropic medium takes the same form as for an isotropic medium. The zero reaction theorem is based on an anisotropic reaction theorem (or modified reciprocity theorem). Conditions under which the anisotropic reaction theorem holds were specified. Generally the complex anisotropic medium is nonreciprocal, but media which satisfy reciprocity were discussed. Finally, mutual and self-impedance elements of a network representation of the complex anisotropic medium of a multiport network were determined using reaction integrals. Impedance element symmetry in regard to the complex anisotropic medium and its complement had been treated.

Integrated circuit technology development has been progressing to a point where use of many dissimilar materials are or will be deposited on top of one another to construct hybrid or monolithic circuits. A comprehensive list of materials used today or potentially usable in the future would be enormous. However, classes of materials can be listed, many of which are anisotropic. They include ferrites, semiconductors (monatomic and compound), dielectrics (isotropic, uniaxial, biaxial), magnetic films, metals, and optically active materials. Awareness of this wide range of materials, and of the need to use methods such as the moment method and spectral domain techniques as discussed in this paper to calculate fields and other microwave and millimeter wave properties of radiators, resonators, waveguides, and junctions, will be essential for future development work.

REFERENCES

An Offset-Fed Reflector Antenna with an Axially Symmetric Main Reflector

DAU-CHYRH CHANG, MEMBER, IEEE, AND WILLARD V. T. RUSCH, FELLOW, IEEE

Abstract—A design method for an offset-fed, dual reflector antenna (Cassegrain type or Gregorian type) system with an axisymmetric main reflector is presented. Geometrical optics (GO) and the geometrical theory of diffraction (GTD) are used to find the surface-current density on the main reflector. A modified Jacobi-Bessel series (JBS) method is used to find the far-field pattern for the physical optics (PO) integral. In the defocused mode of operation, a new technique is developed to find the reflection point on the subreflector corresponding to the defocused feed and a general field point on the main reflector. Two sample systems are designed.

I. INTRODUCTION

The blocking of the cophasal aperture of a reflector by its feed, feed supports, or subreflector generally causes significant RF deterioration of a microwave pencil beam: gain is degraded, beam efficiency is reduced, sidelobes and cross-polarization levels are raised [1], [2]. Consequently, many high-performance, microwave telecommunication and remote-sensing systems have employed offset-fed configurations to eliminate the effects of blocking. Further benefits of an offset-fed feed are a reduction in feed-reflector-feed coupling for a complex feed array, and the additional freedom in choice of supporting structures and the means of thermal control. Many recent communications satellite antennas have had unblocked apertures: RCA Satcom [3], Westar/Intelsat [4], Intelsat IV-A [5], Comstar [6], CS-1 [7], and Intelsat V [8]. The offset principle can also be applied to dual-reflector systems.

This paper describes the study of a new type of unblocked, offset-fed, dual-reflector antenna system (sphere, paraboloid, ellipsoid, etc.) with an axis of symmetry, thus reducing problems of cost and fabrication accuracy. The spherical geometry is particularly attractive from the viewpoint of extreme accuracy because only a single radius of curvature is involved in the fabrication process.

As a potentially useful by-product of its axis of symmetry, and the fact that its focused pencil beam is “cocked” with respect to this axis, a limited-sector conical-scan capability can be achieved without any scan loss by rotation of the feed-subreflector unit about the symmetry axis. Beam scanning and multiple beams can also be generated for different defocused feeds. Thus the system becomes eligible for many space-borne remote sensing applications, requiring conical scanning or beam scanning for high-resolution mapping and target tracking, dual-linear polarization for precise signal discrimination, high beam-efficiency for large target contrast, multiple frequency operation for broad bandwidth and multiple signature for simultaneous wide-angle coverage.

The basic principle of operation of the antenna is shown in Fig. 1: rays from a point source at O are reflected from a convex or concave subreflector, then rereflected from the main reflector as a parallel bundle. The axis of this bundle is “cocked” at an angle \( \alpha \) with respect to the symmetry axis of the reflector \( OV \). As viewed from a direction parallel to the ray bundle, the subreflector lies below the projected aperture of the main reflector (Fig. 1(b)), thus eliminating subreflector aperture blocking.

II. DETERMINATION OF THE SUBREFLECTOR CONTOUR

The main reflector can be any concave axisymmetric surface expressed as (Fig. 1)

\[
z' = f(r^2)
\]