The Applicability of an Inverse Method for Reconstruction of Electron-Density Profiles

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Abstract—Inverse scattering theory is used to reconstruct profiles of electron density from the analytic representation of the reflection coefficient. The complex reflection coefficient $r(k)$ is represented as a rational function of the wavenumber $k$. Using a three-pole approximation for $r(k)$, the one-dimensional inverse scattering theory is applied to obtain a closed-form expression for the electron-density profile function $n(x)$. The integral equation of the inverse scattering theory (Gelfand-Levitan equation) is solved by a differential-operator technique, and several numerical examples are given. Three-pole reflection coefficients are found to be applicable to the reconstruction of relatively thin electron layers which might be generated in the laboratory. Rational reflection coefficients with an increased number of poles are found to be necessary to simulate other electron layers of physical interest. This is demonstrated by comparison of multiple reflection coefficients in the Butterworth approximation with reflection coefficients calculated from Epstein’s direct scattering theory for electron layers. A parameter in the Epstein theory, which characterizes the total electron content of the layer, is related to the number of poles needed to reconstruct that layer. Estimates are thus obtained of the number of poles needed to reconstruct ionospheric layers and other plasmas of physical interest.

INTRODUCTION

A BASIC PROBLEM for many investigators is to determine, via electromagnetic probing signals, the electron concentration $n(x)$ as a function of the penetration distance $x$ in an inhomogeneous plasma. For example, the electron-density profile of the ionosphere can be partially determined by vertical-incidence pulse-sounding techniques. The data base here is the time delay between transmitted and reflected pulses as a function of the carrier frequency. The processing of the data to obtain the electron-density profile by the so-called “virtual height method” has been extensively discussed (e.g., see [1] and [2]). The portion of the electron profile which increases monotonically with penetration distance is reconstructed by the inversion of a group path delay integral which is obtained after using the Wentzel-Kramer-Brillouin (WKB) approximation. A basic limitation of this method is that electron-density values behind maxima of the profile remain unknown. Another method which has been used with some success is to assume a specific model for the electron-density distribution and obtain, by trial-and-error calculations of the optical properties, an empirical fit for a few parameters of this model. This method is limited by the necessity of assuming a specific model profile for $n(x)$.

This paper discusses the applicability of a profile inversion method for simple isotropic lossless one-dimensional plasmas, which is based on an application of the inverse scattering theory. In principle the present method is not subject to the above limitations. This method uses a data base which is the dependence of the complex reflection coefficient $r(k)$ for the plasma on electromagnetic wave frequency at normal incidence, as shown in Fig. 1. The frequency variable will be indicated by the wavenumber $k$, which is defined in terms of the angular frequency $\omega$ as $k = \omega/c$, where $c$ is the velocity of light in vacuum. The electric field amplitude $E(k, x)$ satisfies the wave equation with relative dielectric constant

$$\frac{\varepsilon(k, x)}{\varepsilon_0} = 1 - \frac{q(x)}{k^2}, \quad q(x) \equiv \frac{e^2}{\varepsilon_0 mc^2} n(x). \tag{1}$$

Here $e$ and $m$ are the charge and mass of the electron, and $\varepsilon_0$ is the dielectric constant of free space. For simplicity, the plasma is considered to be confined to the region $x \geq 0$ in Fig. 1. The wave equation can be written as

$$-\frac{d^2}{dx^2} + q(x) E(k, x) = k^2 E(k, x), \tag{2}$$

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where the profile function \( q(x) \) is analogous to the scattering potential in quantum mechanics. Inverse scattering theory [3], [4] provides the relation between the unknown function \( q(x) \) and the reflection coefficient \( r(k) \) through the integral equation

\[
K(x, ct) + R(x + ct) + \int_{-ct}^{x} K(x, \xi) R(\xi + ct) d\xi = 0, \quad -x < ct \leq x, x \geq 0, \quad (3)
\]

where

\[
R(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} r(k) e^{-ikx} dk. \quad (4)
\]

The contour of integration in (4) is understood to be closed by a semicircle in the upper half-plane for \( x < 0 \), and in the lower half-plane for \( x > 0 \). Here the function \( r(k) \) is assumed to be analytic in the upper half of the \( k \) plane (i.e., there are no "bound states" or "trapped modes" [6]). The profile function \( q(x) \) is determined from the solution of (3) through the relation

\[
q(x) = 2 \frac{d}{dx} K(x, x), \quad x \geq 0. \quad (5)
\]

These equations summarize the profile reconstruction method, if the reflection coefficient is given.

It is possible to obtain an analytic solution of (3) for \( K(x, y) \) and the profile function \( q(x) \), if \( r(k) \) can be represented as a rational function of \( k \) [3], [5]–[7]. This is not only a simplification, but it also enables the characterization of plasmas by the pole configuration of \( r(k) \) in the complex \( k \) plane; new insights might be gained from this point of view. It has been noted [6] that the profile functions obtained from three-pole reflection coefficients have shapes which resemble simple ionospheric distributions. This was also indicated by the profile functions for the general three-pole case which were obtained from a Laplace transform method for solving (3) [7]. The three-pole case is rederived here by a differential operator technique [5], [6] and tested for its applicability to the ionosphere in order to clarify the applications of rational representations of \( r(k) \).

The range of applicability of rational \( r(k) \) representations is extended by generalizing to \( N \)-pole reflection coefficients, where the choice of \( N \) depends on the particular problem. Insight into this matter is gained by the comparison of \( N \)-pole reflection coefficients with the reflection coefficients calculated from previous direct scattering theories. (We recall that a direct scattering theory assumes a specific profile and then calculates the resultant scattered fields. An inverse scattering theory, on the other hand, starts from the scattering data and reconstructs the profile function.) Epstein's direct scattering theory [8] is helpful in this regard, since the scattering results are precisely known; it also has been widely used (e.g., see [9]). An estimate of \( N \) is thus obtained for the approximate reconstruction of any of a wide range of electron distributions, which could, in principle, proceed analogously to the three-pole case.

**Inverse Scattering Theory: Three-Pole Case**

The three-pole reflection coefficient is represented as

\[
r(k) = \frac{k_{1} k_{2} k_{3}}{(k - k_{1})(k - k_{2})(k - k_{3})}, \quad (6)
\]

where \( k_{1}, k_{2}, \) and \( k_{3} \) are pole positions in the lower half of the \( k \) plane. We note that \( r(0) = -1 \), as required, and it is also required that \( r(k) = r(-k) \) for real \( k \), where the asterisk means complex conjugate, so that \( R(x) \) in (4) is real. Accordingly, pole positions are chosen as

\[
k_{2} = c_{1} - ic_{2}, \quad k_{1} = -k_{2}^{*}, \quad k_{3} = -ia. \quad (7)
\]

where \( c_{1}, c_{2}, \) and \( a \) are positive real and have the dimensions of inverse length. It will be convenient to use dimensionless quantities, so that in this section all wave vectors are to be normalized by \( a \) (e.g., \( k_{3} = -i \)), and all distances are to be normalized by \( a^{-1} \). Conservation of energy requires \( |r(k)|^{2} \leq 1 \) for all real \( k \); this restricts the domain of allowed pole positions in the complex \( k \) plane [6]. A point in the complex \( k \) plane is represented in Cartesian and polar forms by

\[
k = k' + ik'' = |k| e^{i\theta},
\]

so that the pole position of \( k_{2} \) can be represented as

\[
k_{2} = Ce^{-i\theta}, \quad C = \sqrt{c_{1}^{2} + c_{2}^{2}}, \quad \theta = \arctan (c_{2}/c_{1}).
\]

The conservation-of-energy condition becomes

\[
|k|^{2} + |k|^{2}(1 - 2C^{2} \cos 2\theta) + C^{2}(C^{2} - 2 \cos 2\theta) \geq 0. \quad (8)
\]

The zeros of this expression are

\[
|k|^{2} = (1/2) \left[ -(1 - 2C^{2} \cos 2\theta) \pm \sqrt{D} \right],
\]

where

\[
D = (1 - 2C^{2} \cos 2\theta)^{2} - 4C^{2}(C^{2} - 2 \cos 2\theta)
\]

is the discriminant for the preceding expression. If positive real \(|k|^{2}\) roots are to be avoided, the allowed positions for \( k_{1} \) and \( k_{2} \) poles are restricted to the shaded regions shown in Fig. 2 (with \( a = 1 \) [7]). Region B in Fig. 2 is bounded from above by the curve \( k'' = -1/2 \), and region A is bounded from below by this curve and from above by the curve \(|k| = 2 \cos 2\theta \) in the lower half-plane, as shown in Fig. 2.

A differential operator can be constructed [5], [6] to convert the integral equation (3) to a differential equation. If the
where
\[ r_1 = [(S + T)/2]^{1/2}, \quad r_2 = [(S - T)/2]^{1/2} \]
\[ S = 1 + 2c_2^2 - 2c_1^2, \quad T = \sqrt{(1 - 4c_2^2)(1 + 4c_1^2)} \] (14)
In region A of Fig. 2, T is real, but in region B it is imaginary. Substitution of (13) in (11) yields the three conditions
\[ C_1(x) = 0, \quad A_{-i} = A_i f(r_i)/C_2 \quad (i = 1, 2), \] (15)
where \( f(r_1) \) is given by (9). The three additional conditions required can be obtained from the original integral equation (3).

With the definition \( y \equiv ct \), the procedure is to apply each of the operators \( \partial/\partial y \), and \( \partial^2/\partial y^2 \) to (3), and then set \( y = -x \) in the result. The set of equations obtained by this procedure is
\[
\begin{bmatrix}
1 & G_1 & G_2 \\
0 & r_1 F_1 & r_2 F_2 \\
0 & r_1^2 G_1 & r_2^2 G_2
\end{bmatrix}
\begin{bmatrix}
C_0(x) \\
A_1(x) \\
A_2(x)
\end{bmatrix} = \begin{bmatrix} 0 \end{bmatrix},
\] (16)
where
\[
F_i = F(r_i) = \exp(-r_1x) - \exp(r_1x)f(r_i)/C^2 \quad (i = 1, 2)
\]
\[
G_i = G(r) = \exp(-r_1x) + \exp(r_1x)f(r_i)/C^2.
\] (17)

The solution \( K(x, y) \) given by (13) must satisfy the boundary condition (5), so that the profile function is found to be
\[
q(x) = 4V/w^2,
\]
where
\[
V = \exp(-2r_1x)p_2[2r_2H_1g_2 - H_2[r_2(g_2 + 1)] + r_1(g_2 - 1)] + 2r_1^2[H_1g_2(g_1 + 1) - H_2g_1(g_2 + 2)] + \exp(2r_1x)g_1r_2[2r_2H_1g_2 - H_2g_1(r_1g_2 + 1)] - r_2(g_2 - 1)] + (1 \leftrightarrow 2)
\]
\[
W = (r_2 - r_1) \exp[-(r_1 + r_2)x] + g_2(r_1 + r_2) \exp[-(r_1 - r_2)x] - g_2(r_1 + r_2) \exp[(r_1 - r_2)x] + g_1[r_1 + r_2] \exp[(r_1 + r_2)x]
\] (18)
\[
g_i = f(r_i)/C^2 \quad (i = 1, 2)
\]
\[
H_i = (1/r_i)[f(r_i) - C^2]
\]
\[
= r_i^2 + (1 + 2c_2^2)r_i + c_1^2 + c_2^2 + 3c_2.
\] (20)

The last \( (1 \leftrightarrow 2) \) term in (18) represents three terms which are obtained from the three preceding terms by simply interchanging the subscripts 1 and 2. (This result is equivalent to that previously obtained in [7, eq. (12)], noting that a typographical error has been corrected, since the denominator there should be squared. The calculations in [7] have been confirmed here since the correct denominator was used in the previous computer calculations; the present formulation is cast in a different form and appears to be simpler to evaluate numerically.)

According to (14), the roots \( r_1 \) and \( r_2 \) are real in region A of Fig. 2 and complex in region B. Thus there will be qualita-
ative differences between profile functions corresponding to pole positions in these regions. Fig. 3 shows three cases (q1, q2, q3—solid curves) for pole positions in region A of Fig. 2 and one case (q4—dashed curve) in region B. As the line \( k^2 = 1/2 \) is crossed from above for \( k_1 \) and \( k_2 \) pole positions, the profile changes shape and sign, going from positive to negative. Since profile functions have oscillatory components for pole positions in region B, and furthermore, since all our numerical results in this region exhibit negative profiles, it is plausible that this is true generally. From (1), region B pole positions are thus apparently not relevant for normal incidence. As indicated in [6], negative profile functions are to be considered when analyzing vertically polarized waves which are obliquely incident on an inhomogeneous region. The horizontal and vertical scales of Fig. 3 are normalized to the pole-position parameter \( a \). If we denote dimensional entities by tildes symbols, this scaling is given explicitly by

\[
\tilde{q}(x) = a^2 q(x/a).
\]  

Consider a typical nighttime \( F \) layer, for which the maximum electron density is \( 5 \times 10^5 \) cm\(^{-3} \) and the width of the distribution at half-maximum is 200 km, and attempt to simulate this by \( q_2 \) in Fig. 3 with an appropriate choice of parameters. The width of the layer fixes \( a = (1.65)/200 \approx 8.2 \times 10^{-3} \) km\(^{-1} \), and the maximum electron density associated with \( q_2 \) is 2.66 \( \times 10^{-3} \) cm\(^{-3} \), which is eight orders of magnitude too small for the \( F \) layer. Alternatively, \( a \) can be determined to fit the peak density of the \( F \) layer, but then the width of the distribution is four orders of magnitude too small. It is found that this discrepancy is always encountered, and even worsened, for other pole positions in region A of Fig. 2. The three-pole rational \( r(k) \) expression is therefore not sufficient for the simulation of ionospheric \( F \) layers, and more poles will be required for this purpose.

**APPLICATION OF RATIONAL \( r(k) \) FUNCTIONS**

The wave equation (2) has been solved exactly by Epstein [8] for a class of profile functions, a special case of which

\[
q(x) = 4q_m \exp \left( \frac{x}{H} \right) \left[ 1 + \exp \left( \frac{x}{H} \right) \right]^2
\]  

has been used [10] to model ionospheric layers. Here \( x = x - x_m \), where \( x_m \) is the location of the maximum value \( q_m \) of the profile function, and \( H \) gives the length scale of the distribution. The reflection coefficient from this symmetric profile is [8, 10] (cf. Fig. 1)

\[
r_E(k) = e^{2ikx_m} \frac{\cosh \left( \frac{\pi d_1}{2} \right) \Gamma(iS_1 k/k_1) \Gamma(0.5 - i(S_1 k/k_1 + d_1)) \Gamma(0.5 - i(S_1 k/k_1 - d_1))}{\Gamma(-iS_1 k/k_1)}
\]  

where

\[
d_1 = \sqrt{4S_1^2 - 1/2}, \quad S_1 = 2k_1 H, \quad k_1 = \sqrt{q_m}.
\]

The wave vector \( k_1 \) corresponds to the cutoff frequency for the plasma, i.e., it is the maximum angular plasma frequency divided by \( c \), the velocity of light in free space. The dimensionless parameter \( S_1 \) characterizes the phase thickness or total electron content of the plasma. It is the phase change for the wave at the cutoff frequency in an optical path \( 2H \). The gamma functions of complex argument in (23) can be evaluated by standard techniques [11]. The factor \( \exp \left( 2ikx_m \right) \) arises because of the choice of \( x = 0 \) as the point at which \( r \) is measured as the ratio of reflected to incident electric fields.

A general complex reflection coefficient of the form

\[
r(k) = |r(k)| \exp \left( i\alpha(k) \right)
\]  

can be approximated by (23), or it can be approximated by the rational function

\[
r_N(k) = e^{2ikx_n(-1)^N+1} \frac{k_1 k_2 \cdots k_N}{(k-k_1)(k-k_2) \cdots (k-k_N)},
\]  

where \( N \) is the number of poles which are located in the lower half of the complex \( k \) plane. A proper fit by (23) should at least determine the gross features of the unknown electron distribution, viz., height and width, and so should a fit by (25). In particular, by fitting the approximation (25) to \( r_E(k) \) in (23), it will be possible to relate the number of poles \( N \) to the Epstein profile parameter \( S_1 \). It will then prove possible to use this relation to estimate the number of poles required to reconstruct the gross features of particular electron distributions.

In the rational function approximation of (25) the number of poles \( N \) influences the characteristics of the reconstructed profile, and so do the pole positions, which are restricted by conditions on \( r_N(k) \), which were mentioned in connection with (6). A complete set of conditions were obtained by Kay [3] and are listed in [5]. The phase factor \( \exp \left( 2ikx_n \right) \) in (25) translates the reference point for the reflection coefficient by the amount \( -x_n \) along the \( x \) axis. Equivalently, for the fixed reference point at \( x = 0 \), it translates the profile function associated with (25) for \( x_n = 0 \) by the amount \( x_n \) along the \( x \) axis.

As can be seen from the discussion of the profile reconstruction procedure for the three-pole case, the amount of calculations increases rapidly as \( N \) increases. A particular pole configuration, namely the Butterworth approximation [51], is a reasonably simple representation which provides estimates of the profile functions corresponding to a large number of poles. In this approximation

\[
|r_N(k)|^2 = \frac{1}{1 + k^{2N}}
\]  

with poles only on the unit circle in the lower half-plane.

In (25) for this case

\[
k_j = \exp \left[ -i \left( 1 - \frac{2j - 1}{2N} \right) \pi \right] \quad (j = 1, 2, \ldots, N).
\]  

In order to relate the number of Butterworth poles \( N \) to the Epstein profile parameter \( S_1 \), we will compare the characteristics of the reflection coefficients \( r_E(k) \) and \( r_N(k) \). The bandpass characteristics of \( |r_E(k)| \) depend on \( S_1 \), as shown by the graphs of Fig. 4 which plot \( |r_E(k)| \) versus \( k/k_1 \) for
several \( S_1 \) values. The corresponding phase functions,

\[
\phi(k) = 2kx_m + \pi - \alpha(k),
\]

have been calculated and are plotted in Fig. 5. It is seen in Fig. 4 that the bandpass characteristics of \( |r_E(k)| \) at \( k/k_1 = 1 \) are sharpened as \( S_1 \) increases. Similarly, it is seen from (26) that the bandpass characteristics of \( |r_N(k)| \) are sharpened as \( N \) increases, and \( |r_N(1)|^2 = 1/2 \) in the Butterworth case. In order to compare commensurate values of \( |r_E(k)| \) and \( |r_N(k)| \), the variable \( k \) of (25) must be normalized to \( \kappa_1 = \sqrt{\kappa}/\kappa_m \). The function \( |r_N(k)| \) is plotted in Fig. 6 as a function of the normalized parameter \( k/k_1 \) for the known two- and three-pole Butterworth cases, as well as other three-pole cases for different pole positions in region A of Fig. 2. Curve (2) for the three-pole case in Fig. 6, corresponding to curve \( q_2 \) in Fig. 3, is essentially identical to the three-pole Butterworth case. From Fig. 4 it is evident that, as \( S_1 \) increases, \( |r_E(k)| \) at \( k/k_1 = 1 \) approaches \( 1/\sqrt{2} \) from below, which resembles the behavior of the \( N \)-pole Butterworth approximation. Thus in the Butterworth approximation of (26) and (27) it seems reasonable to replace \( k \) by \( k/k_1 \) for \( N \) large. One would predict that \( q_m \rightarrow 1 \) as \( N \rightarrow \infty \) in this case.

As \( S_1 \) and \( N \) increase, so does the sharpness of the decline of \( |r(k)| \) in the vicinity of \( k/k_1 = 1 \) (cf. Fig. 4). The sharpness of the decline is measured inversely by the size of the interval \( \Delta k/k_1 \) in which it occurs. If this is arbitrarily defined as

\[
\Delta k/k_1 = (k_1 - k_0)/k_1,
\]

Fig. 3. Profile functions for different pole locations in three-pole case;

\( q_1 \), \( c_1 = 0.5 \); \( c_2 = 0.4999 \); \( q_2 \), \( c_1 = 0.8544 \); \( c_2 = 0.4999 \); \( q_3 \), \( c_1 = 0.208 \); \( c_2 = 0.2 \). Negative of profile \( q_4 \) is shown; \( c_1 = 0.8 \); \( c_2 = 0.501 \).

Fig. 4. Reflection coefficient amplitude \( |r(k)| \) versus \( k/k_1 \) for several Epstein profile parameters \( S_1 \).

\( q_4 \) is plotted in Fig. 6 as a function of the normalized parameter \( k/k_1 \) for the known two- and three-pole Butterworth cases, as well as other three-pole cases for different pole positions in region A of Fig. 2. Curve (2) for the three-pole case in Fig. 6, corresponding to curve \( q_2 \) in Fig. 3, is essentially identical to the three-pole Butterworth case. From Fig. 4 it is evident that, as \( S_1 \) increases, \( |r_E(k)| \) at \( k/k_1 = 1 \) approaches \( 1/\sqrt{2} \) from below, which resembles the behavior of the \( N \)-pole Butterworth approximation. Thus in the Butterworth approximation of (26) and (27) it seems reasonable to replace \( k \) by \( k/k_1 \) for \( N \) large. One would predict that \( q_m \rightarrow 1 \) as \( N \rightarrow \infty \) in this case.

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\[
\Delta k/k_1 = (k_1 - k_0)/k_1,
\]
where \(|r(k_0)| \equiv 0.9 \) defines \( k_0 \), then \( S_1 \) for the Epstein case and \( N \) for the Butterworth approximation can each be determined as a function of \( \Delta k/k_1 \), as shown in Fig. 7. The curve for \( S_1 \) is evaluated from (23), and the curve for \( N \) is evaluated from (26), with \( k \) on the right replaced by \( k/k_1 \). Hence, according to the discussion of the preceding paragraph, the curve for \( N \) is expected to be accurate for large \( N \). Also shown in Fig. 6 is a dashed straight-line extrapolation onto the \( N \) curve from the known two- and three-pole Butterworth cases. Hence, by regarding the best estimate of the \( N \) curve as the dashed extrapolation for small \( N \) and the solid portion of the curve for large \( N \), it is possible to form an estimate of the graphical relation between \( S_1 \) and \( N \); the result is shown in Fig. 8.

The number \( N \) of Butterworth poles needed to simulate the \(|r| \) bandpass sharpness characteristics was found to increase with the parameter \( S_1 \), which characterizes the height-width product of the electron distribution. For more complicated distributions, the \(|r| \) bandpass sharpness is expected to be sensitive only to the part of the distribution in the vicinity of its highest peak. The result for \( N \) in Figs. 7 and 8 would depend on the height-width product near this peak. The validity of this remark is seen from the WKB theory for transmission through a potential barrier [12]. The transmission coefficient is given as

\[
|\tau|^2 = 1 - |\tau|^2 = \exp \left( -2k \int_0^s |e(s)/e_0|^{1/2} ds \right),
\]

(30)

where the relative dielectric constant of (1) appears in the integrand, and \( x = a \) and \( x = b \) are the classical turning points, i.e., points at which the relative dielectric constant vanishes. For example, if the profile function is given by

\[
q(x) = \kappa_1^2 \left[ 1 - (x - x_m)^2/\rho^2 \right]
\]

(31)
in the vicinity of its highest peak at \( x = x_m \), then it is found that

\[
|\tau|^2 = \exp \left[ -\pi \kappa_1 \rho (1 - k^2/\kappa_1^2) \right],
\]

(32)

which is expected to be valid for \( T \ll 1 \). The aforementioned dependence of \(|r| \) bandpass sharpness on the height-width product is exhibited by this result. For the Epstein distribution of (22), \( \rho = 2H \), and it is found from (32) that the bandpass sharpness parameter of (29) is given as \( \Delta k/k_1 \approx 0.126/S_1 \) for large \( S_1 \), which agrees quite closely with the \( S_1 \) curve in Fig. 7. Hence, while the number of poles \( N \) found from \(|r| \) bandpass sharpness characteristics in Fig. 7 should be appropriate to model the highest peak of a complicated distribution, it might be insufficient to model the rest of the distribution, which could, for example, be considerably wider than its highest peak.

A good rational function approximation to complex reflection coefficient data should exhibit a closely similar dependence of both amplitude \(|r| \) and phase \( \alpha \) on \( k/k_1 \). While the former depends on the characteristics of the highest peak of a complicated distribution, the latter contains information on the detailed structure of the distribution. In the WKB theory for \( k/k_1 \ll 1 \), the phase is given by [1]

\[
\alpha = -\frac{\pi}{2} + 2k \int_0^a (e(s)/e_0)^{1/2} ds,
\]

(33)

where \( x = a \) is the first classical turning point encountered. The integral term represents the phase change of the wave in a round-trip pass through the optical path between \( x = 0 \) and \( x = a \). If, as an approximation, we regard the profile as being nonnegligible only for \( x > x_1 \), then the part of the phase change associated with optical paths in the profile is

\[
\phi(k) = 2k \int_{x_1}^a (e(s)/e_0)^{1/2} ds.
\]

(34)

It is this part of the phase change which contains information about the structure of the profile. The total variation of \( \phi \) in a frequency sweep from zero up to the cutoff frequency for the plasma is \( \phi(k_1) \), which is evaluated to be on the order of \( \kappa_1 (x_m - x_1) \), where \( x_m \) is the position of the highest peak in the distribution. An evaluation of \( \phi(k_1) \) from (34) for the profile of (31) gives \( \phi(k_1) = \kappa_1 \rho \). For the Epstein profile, which drops off more slowly,

\[
\phi(k_1) = 2S_1 \ln \left[ \cosh \frac{x_m - x_1}{2H} \right].
\]

This result is consistent with the variation of \( \phi \) found for the Epstein case in Fig. 5, in which it is found that \( \phi(k_1) \) is of order \( S_1 \). The calculation of the phase change associated with optical paths in the profile for the rational function
approximation of (25) is the variation of phase from \( k = 0 \) up to \( k > 0 \) with \( x_n = 0 \). This corresponds to \( \phi(k) \) in (34). For the Butterworth case it is found to be a smooth monotonically increasing function which asymptotically approaches the value \( N\pi/2 \) as \( k \) becomes infinite. If the pole positions in (25) are moved away from the Butterworth positions in (27), assuming only that for every pole at \( k_j \) there is one at \(-k_j^*\) (which ensures that \( r(k)^* = r(-k) \)), then the details of the phase variation will change, but the maximum \( \phi \) phase change is still similarly limited by \( N\pi/2 \). Hence, even for quite general pole positions in (25), the number of poles \( N \) required to match the overall phase change \( \phi(k_j) \) for a complicated electron distribution should be of the order of \( k_1(x_m - x_1) \). This is consistent with Fig. 8, where the number \( N \) of Butterworth poles satisfies \( N \approx 3S_1 \).

In the application of a rational \( r(k) \) approximation for a particular case, the procedure adopted would depend on the problem at hand and the information available. It may, for example, be desired to test the rational function approximation for a particular electron layer whose characteristics are known. A first step is to estimate \( S_1 \) by a fit of the Epstein profile in (22). The number of poles \( N \) required for the simulation by (25) can next be estimated from Fig. 8. An attempt could then be made to adjust the pole positions in (25), in conjunction with trial-and-error solutions of the inverse scattering problem, in order to incorporate as many of the known profile characteristics as possible. In principle, the methods used for the three-pole case could be used for this purpose. If, on the other hand, information about the reflection coefficient is known, but the profile function is unknown, the parameter \( cK_1 \) can first be estimated as the cutoff frequency in the \( |r(k)| \) data (e.g., \( |r(k_j)|^2 \approx 0.5 \)). This determines the approximate maximum electron density. The parameter \( \Delta k_1 \) in (29) can next be directly estimated from the \( |r(k)| \) data. The parameter \( S_1 \) determined from Fig. 7 then gives an estimate of \( \rho = 2H \) in (31), which describes the fall-off rate of the distribution in the vicinity of its highest peak. Also found from Fig. 7 is an estimate of the number of poles required to simulate this peak, which may be sufficient for the profile in the case of relatively simple distributions. If, additionally, the phase variation data are processed to extract a value of \( k_1(x_m - x_1) \), the number of poles required to model the entire profile function can be estimated from Fig. 8, with \( S_1 \) replaced by \( k_1(x_m - x_1) \). By manipulating the pole positions to conform to the data in some unspecified way, the approximate profile function is obtained by again solving the inverse scattering problem for the \( N \) pole case.

A simple example of the above considerations is a model [13] nighttime \( F \) layer appropriate to sunspot maximum conditions in the 11-year cycle. This is similar to the example briefly considered at the end of the preceding section. By means of an Epstein fit (cf. (22)) it is found that \( H \approx 50 \) km and \( S_1 \approx 1.28 \times 10^4 \). Hence, from Fig. 8, on the order of \( 4 \times 10^4 \) poles would be required to simulate this \( F \) layer in a rational function \( r(k) \) approximation.

**DISCUSSION**

Procedures have been developed in this paper for the application of a particular inverse scattering theory. In this theory the electron profile of an idealized one-dimensional plasma is reconstructed from a particular representation of the reflection coefficient at normal incidence. The reflection coefficient is represented as a specific rational function of frequency (using the associated free-space wavenumber variable \( k \)) having \( N \) poles. The reconstruction procedure for \( N \) poles was illustrated by the three-pole case, which was solved by the differential operator technique. The three-pole results were found to be inapplicable to electron distributions, such as the ionosphere, which do not have sufficiently small height-width products.

The appropriate choice of \( N \) for a rational approximation to the reflection coefficient \( r(k) \), which is to be used in the reconstruction procedure for a particular plasma application, is facilitated by this investigation. By comparison of the \( |r(k)| \) bandpass sharpness characteristics for Epstein and rational Butterworth reflection coefficients, \( N \) and the Epstein parameter \( S_1 \) were explicitly related; e.g., it was found that \( N \approx 3S_1 \). Despite the simplifying assumptions used in the derivation, it was inferred on the basis of the WKB theory for the reflection coefficient that this relation is quite generally applicable, even for relatively complicated electron distributions. In the event of such a distribution, \( S_1 \) is to be replaced in the above relation by the round-trip phase change \( \Delta \phi \) within the plasma just beneath the plasma cutoff frequency (\( \approx k_1(x_m - x_1) \) in the discussion of the preceding section). This information is contained in the phase variation of the reflection coefficient with frequency. It was also shown that \( |r(k)| \) bandpass sharpness characteristics could be directly related to the characteristics of the distribution in the vicinity of its highest peak. Poles of the rational \( r(k) \) approximation may, of course, deviate from the Butterworth positions, in order to produce improved agreement with the data. This freedom may result in a requirement for fewer poles. However, since the \( \Delta \phi \) variation is limited by \( N\pi/2 \), it is anticipated that the above relation between \( N \) and \( S_1 \) (or \( \Delta \phi \)) gives at least the order of magnitude of the number of poles required to yield the correct height-width product in the reconstructed profile. However, for a distribution with an exceedingly long trailing edge beyond its highest peak, the number of poles would be greater than this. Such a feature should be recognizable in the behavior of \( |r(k)| \) above the plasma cutoff frequency.

Once \( N \) is chosen, and the reflection coefficient is represented as a rational function (e.g., (25)), the solution can proceed analogously to the three-pole case. The \( N \)-pole differential operator is a simple generalization of (9). The \( N \)-pole evaluation of \( R(x) \) from (4) is straightforward, as is the computation of the differential equation corresponding to (12). Solutions of the form \( e^{\tau \alpha} \) will involve \( N \) solutions for \( \tau \), at least one of which will be trivial. The differential equation corresponding to (11) will essentially relate solutions for \( r \) and \( -r \), as in (15). In addition to the initial problem of finding the allowed regions for pole positions in the complex \( k \) plane, a difficult part of the calculation for large \( N \) is finding the solutions for \( \tau \). The solution of the \( N \) equations which correspond to (16) is, however, also an obstacle. Certainly, the feasibility of the rational \( r(k) \) approach is questionable for large \( N \). It is apparently out of the question for the ionospheric \( F \) layer, where \( N \) was found to be on the order of \( 10^4 \). Large \( N \) cases may be more appropriately treated by an alternate solution of the discrete formulation [14], [15] of the inverse scattering problem.

A few poles can be used in the reconstruction of thin electron distributions, such as might be found in certain laboratory plasmas. One attractive candidate is the ultrathin electron (or hole) layer which results from a sharp doping profile in semiconductors by the method of molecular beam
epitaxy [16]. Similarly, thicker, but less dense, electron (or hole) layers arise from ion implantation beams [17]. For a given number of poles, thickness varies inversely as the square root of the electron density. In the above examples, however, the presence of loss due to plasma relaxation processes or collisions must be included. The present inverse scattering theory would have to be extended to include, for example, the complex dielectric constants and profile functions $q(x)$ which arise from these collisions [18].

The detailed methods by which pole positions are adjusted to fit information extracted from reflection coefficient data has not yet been specified. Moreover, the deviations from idealized plasmas which arise from loss processes, magnetic fields, and geometrical edge effects are matters of concern in actual applications.

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REFERENCES


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Arthur K. Jordan (M’59–SM’74), for biography and photograph please see page 189 of this issue.