The Solution to the Lyapunov Equation in Constant Gain Filtering and some of its Applications

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Abstract—Presented in this paper is a solution to the Lyapunov Equation in Constant Gain Filtering. A specific method for deriving the noise reduction ratios for the alpha-beta and alpha-beta-gamma filters are explored using the Lyapunov Equation. This enables us to simplify the computation drastically. For a $2 \times 2$ matrix, one has to solve for one unknown instead of two. For a $3 \times 3$ matrix, one has to solve for three unknowns instead of six and so on. Thus reduction in the number of variables has considerable advantage for higher dimensional filters such as the alpha-beta-gamma filter.

1. Introduction

Sklansky published [8] some of the earliest work on tracking filters used for pointing commands in a tracking radar. This work examined the relationship between the selection of filter coefficients and those coefficients already in use by tracking filters. He proposed such measures of performance as: stability, transient response, noise and maneuver error as functions of the dynamic parameters $\alpha$ and $\beta$. All of the work was based on a frequency domain or z-transform analysis. Benedict-Bordner [1] proposed a relationship between $\alpha$ and $\beta$ based on a pole-matching technique that combined transient performance and noise reduction capability. Analysis performed by Simpson [7], Neal, and Benedict [6] extended this analysis to the $\alpha-\beta-\gamma$ filter. By this time, the Kalman filter was becoming well known in the radar community. Thereafter, the tendency was to discuss the $\alpha-\beta$ and $\alpha-\beta-\gamma$ filters as steady state solution to the Kalman filter. Subsequent papers by a number of authors and Kalata, in particular, derived many results that can be used to characterize tracking performance in a multi-tracking environment. Later, Kalata [5] summarized much of this work in the open literature. Additional work can be found in Gray [3], and in the open literature.

The tracking equations for the $\alpha-\beta$ filter consist of two parts: prediction equations, which are given by

\begin{align*}
x_p(k) &= x_p(k-1) + v_p(k-1)T \\
v_p(k) &= v_p(k-1)
\end{align*}

or

\begin{align*}
X_p(k) &= AX_p(k-1)
\end{align*}

where

\begin{align*}
X_p(k) &= \begin{bmatrix} x_p(k) \\ v_p(k) \end{bmatrix} \\
A &= \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}
\end{align*}

and smoothing equations, which are given by

\begin{align*}
x_s(k) &= x_p(k) + \alpha(x_m(k) - x_p(k)) \\
v_s(k) &= v_p(k) + \frac{\beta}{T}(x_m(k) - x_p(k))
\end{align*}

- $x_s(k) = \text{smoothed position at the k-th interval}$
- $x_p(k) = \text{predicted position at the k-th interval}$
- $x_m(k) = \text{measured position at the k-th interval}$
- $v_s(k) = \text{smoothed velocity at the k-th interval}$
- $v_p(k) = \text{predicted velocity at the k-th interval}$
- $T = \text{radar update interval or period}$
- $\alpha, \beta = \text{filter weighing coefficients}$

The filter equations can be written in matrix form for the $\alpha-\beta$ filter in terms of the smoothed state only as [10]

\begin{align*}
X_s(k) &= F_pX_s(k-1) + G_px_m(k)
\end{align*}
where the matrices are defined as

$$F_\beta = \begin{bmatrix}
1 - \alpha & (1 - \alpha) T \\
-\frac{\beta}{T} & 1 - \beta
\end{bmatrix}$$

(9)

$$G_\beta = \begin{bmatrix}
\frac{\alpha}{T} \\
\frac{\beta}{T}
\end{bmatrix}$$

(10)

For the $\alpha - \beta - \gamma$ filter, we have the smoothing equation

$$X_s(k) = F_\gamma X_s(k-1) + G_\gamma x_m(k)$$

(11)

and

$$F_\gamma = \begin{bmatrix}
1 - \alpha & (1 - \alpha) T \\
-\frac{\beta}{T} & 1 - \beta \\
-\frac{\gamma}{T} & -\frac{\gamma}{T}
\end{bmatrix}
\begin{bmatrix}
\frac{\alpha}{T} \\
\frac{\beta}{T} \\
\frac{\gamma}{T}
\end{bmatrix}$$

(12)

The expected value is explicitly given as

$$P_k = E\{F_\gamma X_s(k-1) + G_\gamma x_m(k)\}$$

(13)

The Kalata relation is obtained from steady state Kalman filter theory assuming zero mean white noise in the position and velocity state equations [4] and is given by

$$\beta = 2(2 - \alpha) - 4\sqrt{1 - \alpha}.$$  

(14)

For the $\alpha - \beta - \gamma$, the commonly used relationship between $\alpha$ and $\beta$ is the Kalata relationship. An additional relation between $\gamma$ and $\alpha - \beta$ is needed. The most common is

$$\gamma = \frac{\beta^2}{2\alpha}$$

(15)

which is known as the Neal-Simpson relation [6].

The update equations for a constant gain filter as a rule can be written in the form

$$X_s(k+1) = FX_s(k) + x_m(k+1)G,$$

(16)

where

$$X_s(k) = n^{th\, \text{vector state measurement}},$$

$$F, G = \text{filter update, gain matrices},$$

$$x_m(k) = \text{scalar measurement model}$$

and where $E(x_m(k)) = 0$. The covariance matrix of a first order system can be defined as [9] (where $'$ denotes transpose)

$$P_k = E\{(X_s(k) - E(X_s(k)))(X_s(k) - E(X_s(k)))'\}$$

(17)

$$= E\{(X_s(k) - X_s(k))(X_s(k) - X_s(k))'\},$$

where $E(X_s(k)) = X_s(k) = F\bar{X}_s(k - 1)$. One can calculate $P_k$ as

$$P_k = E\{\eta_k\eta_k' + \sigma^2_m G'G\}.$$  

(18)

The expected value is explicitly given as

$$P_k = FP_{k-1}F' + \sigma^2_m G'G',$$

(19)

where $E(x_m(k)x_m(k)) = \sigma^2_m(k)$. In steady state ($P_k = P_{k-1} = P$), so the covariance is in the form of a Lyapunov matrix equation

$$FPF' - P = -\sigma^2_m G'G',$$

(20)

which can be written as ($M = \sigma^2_m G'G$)

$$P = FPF' + M.$$  

(21)

This is a Lyapunov equation.

II. Solution to the Lyapunov equation

An approach [2] to solving a Lyapunov equation ($F, M$ are known) is to define an anti-symmetric matrix $S$ as

$$S = PF' - FP.$$  

(22)

For a $2 \times 2$ matrix, one has to solve for one unknown instead of the original equation since it is antisymmetric as is the matrix; $B = FM - MF'$. Now add these two equations together and after a few manipulations, we have a solution for $P$, namely that it is a solution to the equation

$$FSP' - S = FM - MF' = B.$$  

(23)

which is another Lyapunov equation with one fewer unknowns instead of the original equation since it is antisymmetric as is the matrix; $B = FM - MF'$. Now add these two equations together and after a few manipulations, we have a solution for $P$, namely that it is a solution to the equation

$$P = (F + I)^{-1}(S - M)(F^3 - I)^{-1}.$$  

(24)

Since the matrices $P$ and $M$ are symmetric, the matrices $S$ and $B$ are anti-symmetric ($S = -S'$, $B = -B'$). The fact that $S$ is anti-symmetric simplifies our calculations for $P$ can be considerably simplified by solving for Lyapunov equation for $S$ versus solving the original equation for $P$. For a $2 \times 2$ matrix, one has to solve for one unknown instead of two; for a $3 \times 3$ matrix, one has to solve for three unknowns instead of six, etc. Thus reduction in the number of variables has considerable advantage for higher dimensional filters such as the $\alpha - \beta - \gamma$ filter.

Without loss of generality, we can assume that $S$ and $B$ can be written as

$$S = s \begin{bmatrix}
0 & -1 \\
1 & 0
\end{bmatrix},$$  

(25)

and

$$B = b \begin{bmatrix}
1 & -1 \\
0 & 1
\end{bmatrix}.$$  

(26)
The matrix $F$ will be written in a general form as

$$F = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix}$$

therefore,

$$FSF^* = \begin{bmatrix} f_{11} & f_{12} \\ f_{21} & f_{22} \end{bmatrix} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix} \begin{bmatrix} f_{11} \\ f_{21} \end{bmatrix} = s \text{Det}(F) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

(\text{Det}(F) $\neq$ determinant of $F$) so that we have

$$FSF^* - S = s(\text{Det}(F) - 1) \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix},$$

which implies

$$s = b(\text{Det}(F) - 1)^{-1},$$

so $S$ is

$$S = b(\text{Det}(F) - 1)^{-1} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}. $$

For example, if we apply this method to the $\alpha - \beta$ filter our computations are simplified. First, recall that

$$F = \begin{bmatrix} 1 - \alpha & (1 - \alpha)T \\ -\frac{\delta}{T} & 1 - \beta \end{bmatrix}$$

$$\text{Det}(F) = 1 - \alpha$$

and

$$G = \begin{bmatrix} \frac{\alpha}{T} \\ \frac{\beta}{T} \end{bmatrix}.$$ 

By substituting $F$ and $G$ into the Lyapunov equation for $S$, the right hand side becomes

$$B = (FGG' - GG'F) \sigma_n^2 = \frac{\sigma_n^2 \rho^2}{T} \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix}.$$ 

Now $s$ is

$$s = \frac{b}{-\alpha} = \frac{\sigma_n^2 \rho^2}{T \alpha}$$

Solving for $S$, yields

$$S = \frac{\sigma_n^2 \rho^2}{T \alpha} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}.$$ 

One can now solve for $P$ by substituting $S$ into it and noting

$$(F + I)^{-1} = \begin{bmatrix} \frac{2 - \beta}{4 - 2 \alpha - \beta} & -\frac{T(1 - \alpha)}{4 - 2 \alpha - \beta} \\ \frac{2 - \beta}{4 - 2 \alpha - \beta} & \frac{T(1 - \alpha)}{4 - 2 \alpha - \beta} \end{bmatrix}$$

(note $\text{Det}(R) = 1 - \alpha$) and

$$A = (I - GH) \Psi_1 \sigma_n^2 \Psi_1' (I - GH)' + K \sigma_n^2 K'$$

$$= \sigma_n^2 \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix} + \sigma_n^2 \begin{bmatrix} \frac{\alpha^2}{T} & \frac{\beta}{T} \frac{\beta}{2} \end{bmatrix}.$$
Without loss of generality, we can take $\sigma_n^2 = 0$, so

$$A = \sigma^2_w \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$$

Now

$$B = AR' - RA = \sigma^2_w (1-\alpha) T \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

$$S = -\frac{\sigma^2_w (1-\alpha) T}{\alpha} \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}$$

The error covariance matrix can be written

$$P = (R + I)^{-1}(S - A)(R' - I)^{-1}$$

$$= \sigma^2_w \begin{bmatrix} T^1_z(0) & T^1_v(0) \\ T^2_z(0) & T^2_v(0) \end{bmatrix}$$

So the noise reduction ratio for the position and the noise reduction ratio for the velocity are given as before while the transient noise reduction ratios are

$$T^1_z(0) = \frac{(2-\alpha)(1-\alpha)^2 T^2}{\beta D}$$

$$T^1_v(0) = \frac{\alpha^2(2-\alpha) + 2\beta(1-\alpha)}{\beta D}$$

$$T^2_z(0) = \frac{(1-\alpha)(\beta + 2\alpha - \alpha^2) T}{\beta D}$$

Note, this summarizes the standard approach taken in the literature; however the number of unknowns one has to solve for can be reduced considerably by using this method.

Another approach to maneuver uncertainty is to incorporate it into both the position and velocity prediction components in matrix form (which we refer to as Model 2) where

$$\Psi_2 = \begin{bmatrix} T^2/2 \\ T \end{bmatrix}$$

and the other matrices are the same as before.

$$P = \sigma^2_a \begin{bmatrix} T^2_z(0) & T^2_v(0) \\ T^2_z(0) & T^2_v(0) \end{bmatrix}$$

For Model 2, the noise reduction ratio for the position and the noise reduction ratio for the velocity are given as before while the transient reduction ratios are

$$T^2_z(0) = \frac{(1-\alpha)^2 T^2}{2\alpha \beta}$$

$$T^2_v(0) = \frac{T^2}{4} \left[ \frac{2\alpha^2 - 3\alpha \beta + 2\beta}{\alpha \beta} \right]$$

(Note these results were first obtained by the traditional means by W. J. Murray)

For the three state $\alpha - \beta - \gamma$ filter, we can assume without loss of generality, $S$ can be written as

$$S = \begin{bmatrix} 0 & s_{12} & s_{13} \\ -s_{12} & 0 & s_{23} \\ -s_{13} & -s_{23} & 0 \end{bmatrix}$$

while at the same time $B$ is known with the form (e.g. we know the $b_{ij}$)

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ -b_{12} & 0 & b_{23} \\ -b_{13} & -b_{23} & 0 \end{bmatrix}$$

The matrix $F$ is known and can be written in a general form as

$$F = \begin{bmatrix} f_{11} & f_{12} & f_{13} \\ f_{21} & f_{22} & f_{23} \\ f_{31} & f_{32} & f_{33} \end{bmatrix}$$

The following observation is useful in general, namely that if

$$FSF' - S = B$$

and both $B$ and $S$ are antisymmetric, then so is $FSF'$ is also antisymmetric; thus we need only concern ourselves with the upper triangular portion of the matrix. Thus the product $FSF'$ reduces to the matrix

$$FSF' = \begin{bmatrix} 0 & u_{12} & u_{13} \\ -... & 0 & u_{23} \\ -... & -... & 0 \end{bmatrix}$$

where

$$u_{12} = -(f_{12}s_{12} + f_{13}s_{13}) f_{21} + (f_{11}s_{12} - f_{13}s_{23}) f_{22} + (f_{11}s_{13} + f_{12}s_{23}) f_{23}$$

$$u_{13} = -(f_{12}s_{12} + f_{13}s_{13}) f_{31} + (f_{11}s_{12} - f_{13}s_{23}) f_{32} + (f_{11}s_{13} + f_{12}s_{23}) f_{33}$$

$$u_{23} = -(f_{22}s_{12} + f_{23}s_{13}) f_{31} + (f_{21}s_{12} - f_{23}s_{23}) f_{32} + (f_{21}s_{13} + f_{22}s_{23}) f_{33}$$

Since $B$ is also anti-symmetric, it has the form

$$B = \begin{bmatrix} 0 & b_{12} & b_{13} \\ -... & 0 & b_{23} \\ -... & -... & 0 \end{bmatrix}$$

and solving the Lyapunov equation for $S$ reduces to solving the three equations

$$b_{12} = -(f_{12}s_{12} + f_{13}s_{13}) f_{21} + (f_{11}s_{12} - f_{13}s_{23}) f_{22} + (f_{11}s_{13} + f_{12}s_{23}) f_{23} - s_{12}$$

$$b_{13} = -(f_{12}s_{12} + f_{13}s_{13}) f_{31} + (f_{11}s_{12} - f_{13}s_{23}) f_{32} + (f_{11}s_{13} + f_{12}s_{23}) f_{33} - s_{13}$$
\[ b_{23} = -(f_{23}s_{12} + f_{23}s_{13})s_{31} \]
\[ + (f_{21}s_{12} - f_{23}s_{23})s_{32} \]
\[ + (f_{21}s_{13} + f_{23}s_{23})s_{33} - s_{23} \] (76)

in the three unknown \( s_{ij} \). These three equations can be written
\[ (A - I)S_1 = B_1 \] (77)

where
\[ S_1 = \begin{bmatrix} s_{12} \\ s_{13} \\ s_{23} \end{bmatrix} \] (78)
\[ B_1 = \begin{bmatrix} b_{12} \\ b_{13} \\ b_{23} \end{bmatrix} \] (79)

and
\[ A = \begin{bmatrix} f_{11}f_{22} - f_{12}f_{21} & f_{11}f_{23} - f_{13}f_{21} & f_{12}f_{23} - f_{22}f_{13} \\ f_{21}f_{31} + f_{11}f_{32} & f_{13}f_{31} + f_{11}f_{33} & f_{12}f_{33} - f_{22}f_{13} \\ f_{23}f_{31} - f_{22}f_{31} & f_{23}f_{33} - f_{22}f_{33} & f_{23}f_{33} - f_{22}f_{33} \end{bmatrix} \] (80)

Thus the solution for \( S_1 \) is
\[ S_1 = (A - I)^{-1}B_1 \] (81)

which gives the components of \( S_1 \) entirely in terms of known quantities.

One can now solve for \( P \) by substituting \( S \) into it and noting,
\[ P = (F_\gamma + I)^{-1}(S - \sigma_n^2 G_x G_y')(F_\gamma^T - I)^{-1} \] (82)
\[ = \sigma_n^2 \begin{bmatrix} P_z(0) & P_{vx}(0) & P_{x}(0) \\ P_{vx}(0) & P_{xx}(0) & P_{v}(0) \\ P_{x}(0) & P_{v}(0) & P_a(0) \end{bmatrix} \]

The elements of this covariance matrix reveal the noise reduction ratios for position \( (P_z) \), velocity \( (P_v) \), acceleration \( (P_a) \), position-velocity \( (P_{px}) \), and velocity-acceleration \( (P_{va}) \). These results are well known, so we don't replicate them. The algebra for obtaining them is simpler using this method. For the transient case, there are three models that are possible to consider:

\[ \Psi_1 = \begin{bmatrix} T^3/3! \\ 0 \\ 0 \end{bmatrix} \] (83)
\[ \Psi_2 = \begin{bmatrix} T^3/3! \\ T^2/2 \\ 0 \end{bmatrix} \] (84)
\[ \Psi_3 = \begin{bmatrix} T^3/3! \\ T^2/2 \\ T \end{bmatrix} \] (85)

To our knowledge, these cases remain to be solved and would lead to some useful results. We will address these in a future publication.

III. Conclusion

Note this is a variation on work by [9]. Through the process of matrix reduction, we were able to simplify the calculations significantly when solving for anti-symmetric matrix \( S \). Since \( S \) and \( B \) are anti-symmetric, the number of unknowns to solve for in the \( \alpha - \beta - \gamma \) filter were cut in half.

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