A More Complete Analysis for Subnyquist Band-Pass Sampling

Abdon De Paula and Ron J. Pieper
Dept. of Electrical and Computer Engineering
Naval Postgraduate School
Monterey, CA 93943

Abstract

One established method for analyzing the minimum sampling frequency of band-pass signals is based on a quadrature representation for such signals. By modifying this method it is possible to show that the allowed range in sampling frequencies is not simply greater than the minimum predicted. The allowed sampling frequencies lie in bands up to the Nyquist limit.

Introduction

Undergraduate electrical engineering students are commonly exposed to the baseband sampling in their communication course(s). The constraint on the minimum sampling is covered by the Nyquist sampling theorem. The theorem dictates that the minimum sampling rate is equal to twice the highest frequency present in the signal. This frequency will not in general be the minimum for band-pass signals. Fairly theoretical treatments of sampling frequency general limitations can be found in the literature [1]. Most undergraduate texts introduce this concept by showing a plot of the minimum sampling frequency versus the highest frequency in the band-pass signal without very much analysis. [2,3]. At least one text [4] presents an intuitive analysis of this phenomena based on the quadrature representation for band-pass signals. A part of the overall proof the author shows that exact reconstruction can be performed if the ratio of the highest frequency in the defined signal to the band-pass bandwidth satisfies the condition of being an integer. Through graphical considerations, corresponding to defining a new center frequency and a new bandwidth, the author [4] shows that this concept can be applied to an arbitrary band-pass signal. Nonetheless this approach, only generates the minimum sampling frequency and does not deal with other significant limitations on the sample rate. By modifying the established method it is possible to show that the allowed range in sampling frequencies is not simple greater than the minimum predicted. The allowed range actually lies in bands, up to the Nyquist limit, which are readily predicted after making some simple enhancements to the original approach. These results are in agreement with similar curves [5] derived by an alternate scheme. The emphasis of the paper to be presented would be on the graphical modifications to the original approach which lead to these conclusions.

Background

We start the analytical development with a statement for reconstruction [4] of the band-pass signal, g(t), from its samples.

\[ g(t) = \sum_{n=-\infty}^{\infty} g(nT_s) \text{sinc}(2B'(t-nT_s)) \times \cos(2\pi f_c'(t-nT_s)) \]  

(1)

where the sample period, i.e 1/T_s, should satisfy,

\[ T_s = \frac{1}{4B'} \]  

(2)
and it is also required that

\[
\frac{f_c + B'}{2B'} = \text{INTEGER}.
\]

The discussion to follow concentrates on providing the correct guidelines for evaluating \(f_c'\) and \(B'\).

The salient characteristics of the spectrum \(G(f)\) of the band past signal \(g(t)\) are represented on Figure 1. For the special case that

\[
\frac{f_{\text{max}}}{2B} = \text{INTEGER}
\]

the reconstruction rule is valid with \(f_c' = f_c\) and \(B' = B\). Note this is consistent with (3). In order to keep this report reasonable brief the reader is referred to a clear presentation of this proof in the education literature [4]. It is also shown in this work that the reconstruction theorem can be extended to arbitrary band-pass signals not meeting condition (4). This is done by introducing parameters \(B'\) and \(f_c'\) which will satisfy (3). However previous emphasis has been on predicting the minimum sampling frequency with out considering other relevant limitations. In the next section we generalize the original approach [4] to obtain a more complete description of these limitations.

Analysis

The modified approach begins with the general statement,

\[
i_{\text{max}} \leq \frac{f_{\text{max}}}{2B} < i_{\text{max}} + 1,
\]

valid for an arbitrary band-pass signal. It follows that:

\[
i_{\text{max}} = \text{INTEGER} \left\lfloor \frac{f_c + B}{2B} \right\rfloor.
\]

It is worth noting that \(2(i_{\text{max}} - 1)\) is the maximum number of spectral aliases of \(G(f)\) that can be fit, without overlap, between \([f_c - B, f_c + B]\).

The first option, as previously explored [4], considers the extension of the bandwidth as shown in Fig 2b relative to Fig 2a. This case, as represented in Fig 2b, will be shown to correspond to the lower limit of on allowed band of frequencies defined by:

\[
f_{s1} = 4 \times B' \quad \text{(7a)}
\]

as predicted by (2). It should also be possible, consistent with guideline (3), to allow on extension of the bandwidth on the opposite side as represented in Fig. 2c. The associated sampling frequency

\[
f_{su} = 4 \times B' \quad \text{(7b)}
\]

will be shown to correspond to the upper edge of the allowed band of sampling frequencies. Between these limits (7a) and (7b) a more general representation, Fig. 2d, is possible. From \(\Delta f_R\), defined on Fig. 2d, and examination of Fig. 2a, it follows that condition (3) can be re-expressed as:

\[
\frac{f_c + B + \Delta f_R}{2B'} = i,
\]

where \(i\) is a positive integer less than or equal to \(i_{\text{max}}\). In order to cover all cases of Fig 2 we can define

\[
\Delta f_R = 2 \times (B' - B) x,
\]

where \(x\) is allowed to range from \([0,1]\). Note \(x = 0\) and \(x = 1\) corresponds to Fig 2b and Fig 2c respectively. After substitution of (9) into (8) and solving for \(B'\) leads to:
\[ B' = \frac{f_c + B(1-2x)}{2i-2x}, \quad (10) \]

and therefore from (2)

\[ \frac{f_s}{B} = 4B' = 4 \frac{f_c + (1-2x)}{2i-2x} \quad (11) \]

From (11) the upper band limit

\[ \frac{f_{su}}{B} = 4 \frac{f_c - 1}{2(i-1)} \quad (12a) \]

is obtained by setting \( x = 1 \). For the lower limit

\[ \frac{f_{s1}}{B} = 4 \frac{f_c + 1}{2i} \quad (12b) \]

again obtained from (11) but after setting \( x = 0 \).

It is already established that \( i_{max} \) satisfies (6). Consistent with the brief discussion following (6) it is correct to interpret \( 2(i - 1) \) as the number of aliases generated without overlap between \( [f_c - B, -f_c + B] \). Ostensibly the minimum \( i \) should be 2 guaranteeing at least one pair of aliases between \( [f_c - B, -f_c + B] \). This is the correct conclusion for the upper limit (12a) however the minimum \( i \) can, for the lower limit (12b), be extended down to \( i = 1 \). The special case \( i = 1 \) corresponds to no aliases produced between \( [f_c - B, -f_c + B] \). Expression (12b) then reduces to the standard Nyquist statement: \( f_c = 2f_{max} \). It is then evident that for \( i = 1 \) there should be no upper limit.

Following a more standard convention [2] the lower edge frequency defined as,

\[ f_o = f_c - B \quad (13) \]

which is shown on Fig 1, is used as the independent variable for plotting. In addition if we define the number of aliases as

\[ N = 2(i-1) \quad (14a) \]

It follows from (6), after letting \( N_{max} = 2\{(i_{max} - 1) \}

\[ N_{max} = 2 \times \text{INTEGER} \left( \frac{f_o}{2B} \right) \quad (14b) \]

Equation (12a) can now be transformed to

\[ \frac{f_{su}}{B} = 4 \left( \frac{f_o}{B} \right) \frac{1}{N}, \quad \frac{f_o}{B} \geq 2 \quad (15a) \]

where \( N = 2, 4, ... N_{max} \) and (12b) becomes

\[ \frac{f_{s1}}{B} = 4 \left( \frac{f_o + 2}{B} \right) \frac{1}{N+2}, \quad \frac{f_o}{B} \geq 2 \quad (15b) \]

where \( N = 0, 2, 4, ... N_{max} \). Fig 3 compactly represents these results (15), which are in complete agreement with curves obtained by an alternate scheme [4]. As stated in the text the curves generated from (15a) lie above the curves generated from (15b) for the same \( N \geq 2 \). Therefore taking \( x = 1 \) and \( x = 0 \), as shown in Fig 2, correctly dictated the upper and lower band edges respectively.

We note that it is not correct to assume that all frequencies above the minimum \( (f_o/B) \), shown as a dark line on Fig 3, will permit reconstruction (2). It is also clear from Fig. 3 that the allowed range of sampling frequencies falls into bands which increase in number as \( f_o/B \) increases.

For completeness we point out that the physical condition on \( f_o ' \) is that it divides the modified bandwidth, \( 2B' \) in half. After reference to (3) and (10) it can be shown after some algebra that:
and therefore the reconstruction formula is defined within the bands.

Conclusion

These results show that it is not sufficient to indiscriminately sample band-pass signals at frequencies greater than the minimum usually presented in communication courses. The allowed frequencies for sampling were found to lie in bands up to the Nyquist limit. The simple extension of the originally established approach which leads to these conclusions can readily be appreciated by the engineering communications student.

References


SUB-NYQUIST SAMPLING FOR BANDPASS SIGNAL

NYQUIST REGION

N=2
N=4
N=6
N=8

Figure 3 Allowed Sampling Bands