Describing spatiotemporal couplings in ultrashort pulses using amplitude coupling coefficients

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Abstract—An amplitude coupling coefficient has been defined to quantify spatiotemporal couplings in ultrashort pulses, which is dimensionless and normalized. The amplitude couplings in ultrashort pulses, such as first-order spatial chirp and angular dispersion caused by angular dispersion elements have been studied by using the defined amplitude coupling coefficient, and corresponding amplitude coupling coefficients have been given analytically. Furthermore, the second-order spatial chirp caused by angular dispersion elements have been studied.

Keywords- Spatiotemporal couplings; Ultrashort pulse; Spatial chirp; Angular dispersion

I. INTRODUCTION

In the design of optical systems using broadband optical pulses, the manipulation of spatiotemporal couplings is one of the most important issues [1]. The diffraction and refraction of broadband optical pulses propagating through the systems is usually frequency dependent, leading to a coupling of spatial and temporal effects. For ultrashort pulses comprising only a few oscillations, a coupling of spatial and temporal effects becomes important even for propagation in a nondispersive medium [2-3]. Making use of spatiotemporal couplings of ultrashort pulses allows some applications, such as pulse compression [4] and shaping [5]. They are not possible to be realized in their absence. However, spatiotemporal couplings can also be detrimental, for example, when focusing an ultrashort pulse [6]. For these reasons, much attention has been paid to spatiotemporal couplings in recent years.

To date, many researchers have tried to develop a common quantity to describe or quantify spatiotemporal couplings. Unfortunately, such a general quantity does not currently exist. In [1], a degree of spatiotemporal uniformity was defined to quantify spatiotemporal couplings in both amplitude and frequency domains. In [5], spatiotemporal couplings in femtosecond pulse shaping were quantified by the space-time coupling constant \( \Delta \Omega / \Delta r \) or the frequency-to-space mapping \( \Delta \Omega / \Delta r \). In [7], Akturk presented a general theory of first-order spatiotemporal couplings, and classified spatiotemporal couplings into amplitude couplings and phase couplings. Amplitude couplings include spatial chirp [8], angular dispersion [9], pulse-front tilt [10], etc. Phase couplings include wave-front rotation, wave-front-tilt dispersion, angular spectral chirp, etc. In [11] and [12], several correlation coefficients were defined to evaluate the severity of first-order and second-order amplitude couplings, respectively. However, to our knowledge, there is a lack of literatures about how to quantify arbitrary order amplitude couplings.

In this paper, we define an amplitude coupling coefficient, which can be used to quantify arbitrary order amplitude couplings in ultrashort pulses. With the amplitude coupling coefficient, we study the amplitude couplings in ultrashort pulses, such as spatial chirp and angular dispersion caused by angular dispersion elements.

II. DEFINITION OF THE AMPLITUDE COUPLING COEFFICIENTS

In this paper, we mainly consider amplitude couplings in the \( x-\omega \) and \( k-\omega \) domain. Extension to the \( x-t \) and \( k-t \) domains is straightforward. In the \( x-\omega \) domain, the amplitude couplings are spatial chirp. The electric field of the pulse in the \( x-\omega \) domain can be expressed in the form

\[
E(x, \omega) = |E(x, \omega)| e^{i\Phi(x, \omega)}. \tag{1}
\]

where \( |E(x, \omega)| \) denotes the spectral amplitude, and \( \Phi(x, \omega) \) is the spectral phase. In [1], Dorrer et al. proposed the degree of spatiotemporal uniformity \( \rho \) to quantify spatiotemporal couplings in both amplitude and phase, i.e.,

\[
\rho = \frac{\iint d\omega d\omega \int |E(x, \omega)|^2 d\omega}{\iint |E(x, \omega)|^2 d\omega d\omega} \tag{2}
\]

The case of \( \rho = 1 \) corresponds to the absence of space–time coupling. However, \( \rho \) is not suitable for describing amplitude couplings such as spatial chirp or angular dispersion, which contains couplings only in amplitude.

Considering that the amplitude couplings depend only on \( |E(x, \omega)| \), the amplitude coupling coefficient \( \mu_a \) can be defined as

\[
\mu_a = \frac{1}{\iint |E(x, \omega)|^2 d\omega d\omega} \left( \iint |E(x, \omega)|^2 d\omega d\omega \right)^2 \tag{3}
\]

In (3), \( \mu_a \) is normalized and dimensionless. \( \mu_a = 0 \) corresponds to a pulse free of amplitude couplings, while \( 0 < \mu_a < 1 \) indicates that some amplitude couplings are present. Moreover, an
increased value of $\mu_a$ indicates an increase in the magnitude of amplitude couplings.

As shown in Fig. 1(a), the “spatial dispersion (SPD) [8]” is zero, i.e., $\partial x_0/\partial \omega = 0$, where $x_0$ is the mean beam position for a given frequency $\omega$ whereas SPD gets a nonzero value in Fig. 1(b) and a maximum value in Fig. 1(c). Thus, an increased value of $\mu_a$ indicates an increase in the magnitude of spatial chirp. Analogous quantities can be defined for the other amplitude couplings (in the $k$-t, $x$-t, and $k$-t domains).

III. SPATIAL CHIRP

In the $x$-$\omega$ domain, the amplitude couplings are spatial chirp [7]. The schematic diagram of spatial chirp can be seen clearly in Fig. 2. As shown in Fig. 2, the refraction of broadband optical pulses propagating through the prism pair is frequency dependent, which leads to an amplitude coupling in the $x$-$\omega$ domain, i.e., spatial chirp.

Gaussian pulses with Gaussian spatial profiles with first-order spatiotemporal couplings in the $x$-$\omega$ domain can be expressed in the form [7]

$$E(x,\Omega) = \exp(R_{x\omega}x^2 + 2R_{x\omega x\omega}x\Omega - R_{\omega\omega}\Omega^2) \exp(Ax^2 + 2Bx\Omega - C\Omega^2) \times \exp[i(Ex^2 + 2Fx\Omega - G\Omega^2)].$$

In (4), $A=R_{x\omega}^2$, $B=R_{x\omega}^2$, $C=R_{x\omega x\omega}^2$, $E=R_{x\omega}x\omega$, $F=R_{x\omega}x\omega$, $G=R_{\omega\omega}$, $\Omega=\omega\omega_0$, and $\omega_0$ is the central angular frequency, where we have used a shortcut notation, superscript “R” and “I”, for the real and image part of the parameter, respectively. $B$ and $F$ are the coupling coefficients for amplitude couplings and phase couplings, respectively. Because $A$ and $C$ are related to beam spot size and bandwidth, respectively [7], $A$ is always negative, whereas $C$ is always positive. Introducing (4) into (3), the amplitude coupling coefficient for an ultrashort pulse described by (4) can be deduced as

$$\mu_a = 1 - \left(\frac{B^2 + AC}{AC}\right)^{1/2}. \quad (5)$$

For a monochromatic light beam ($C \to \infty$), equation (5) leads to $\mu_a = 0$, indicating that the amplitude couplings do not exist for a monochromatic light Gaussian beam. For the case of the amplitude coupling coefficient $B = 0$, there is $\mu_a = 0$, indicating that the amplitude couplings do not exist.

The dispersive nature of an angular dispersion component can be described approximately by Martinez’s model [13]. By using the Kirchhoff–Fresnel integral, an expression of the electric field of Gaussian pulses with Gaussian spatial profiles at an arbitrary distance from an angular disperser can be expressed in the frequency domain as [13,14]

$$E(x,\Omega) = h \exp(-\tau^2/4) \exp(-ikz^2/2z) \times \exp\left[\frac{ikz}{2z} \frac{q(d)}{q(d + \alpha^2 z)}(x + \beta^2 z)^2\right],$$

where $h_3$ is an amplitude constant, $\tau$ the pulse parameter related to the pulse width, $k_3$ the wavenumber of the central frequency, $d$ the distance from the Gaussian beam waist to the angular dispersion component, $z$ the propagation distance of the laser beam from angular dispersion component (see Fig. 1 in [15]), and $q$ a complex parameter of the Gaussian beam. $\alpha = \partial \theta/\partial \gamma$ is the factor of angular magnification of the beam, where $\theta$ is the emerging angle and $\gamma$ the angle of incidence. $\beta = \partial \theta/\partial \alpha$ is the first-order angular dispersion factor of the angular dispersion component.

It is easy to reduce (6) to be the form of (4), and corresponding parameters $A$–$C$ can be obtained, i.e.,

$$A = -\frac{kz_3\alpha^2}{2\left(d + \alpha^2 z^2\right)^2 + z_3^2},$$

Fig. 1. Profiles of an ultrashort pulse with increasing amounts of spatial chirp, and hence with increasing values of $\mu_a$, where $\Omega = \omega - \omega_0$, and $\omega_0$ is the central angular frequency. (a) $\mu_a=0.00$. (b) $\mu_a=0.40$. (c) $\mu_a=0.80$.  

Fig. 2. Spatial chirp after a prism pair.
\[ B = -\frac{k_z \alpha^2 \beta z}{2 \left( d + \alpha^2 z \right)^2 + \frac{r^2}{z^2}} \]  
\[ C = \frac{k_z \alpha^2 \beta z}{2 \left( d + \alpha^2 z \right)^2 + \frac{r^2}{z^2}} \]  
\[ \text{where } z_R = \frac{\pi}{\beta_0} \text{ is the Rayleigh length of the Gaussian beam, } s \text{ beam waist, and } \lambda_0 \text{ the central wavelength.} \]

Introducing (7)-(9) into (5), the amplitude coupling coefficient for the first-order spatial chirp caused by an angular disperser can be written as

\[ \rho_1 = 1 - \frac{r^2}{2} \]  
\[ \text{(10)} \]

Equations (9) and (10) imply that first-order spatiotemporal couplings caused by an angular disperser depend on the angular dispersion parameters \( \rho_1, \beta_0 \), the pulse width parameter \( r \), the Rayleigh length \( z_R \), and the propagation distance \( z \) intimately. It follows from (9) that \( \partial C/\partial z \geq 0 \). Then, it can be readily shown from (10) that with the increase of propagation distance, the amplitude couplings increase. These properties can also be seen clearly in Fig. 3. As shown in Fig. 3, at a small propagation distance from the disperser \( z \ll z_R \), \( \mu \approx 0 \). This can be explained by using (8). The amplitude coupling coefficient \( B=0 \) at a small propagation distance, resulting in \( \mu \approx 0 \). Figure 3 also shows that at a large propagation distance \( z \gg z_R \), \( \mu \) tends to zero. This can be explained according to (10). For the case of \( z \rightarrow \infty \), \( \mu \) tends to a maximum value 0.994. From another point of view, because the spectral lateral walk-off increases with propagation distance and finally tends to constant (see Fig. 2 in [15]), the amplitude couplings (i.e. spatial chirp) also increase with propagation distance and finally remain unchanged.

It is also interesting to compare the amplitude coupling coefficient \( \mu_a \) with another normalized parameter proposed for describing the first-order spatial chirp, i.e., the correlation coefficient \( \rho_{\text{cor}} \) [7,11]. An increased value of \( \rho_{\text{cor}} \) indicates an increase in the magnitude of first-order spatial chirp. The correlation coefficient \( \rho_{\text{cor}} \) is not very sensitive to the large amounts of spatiotemporal couplings. As shown in Fig. 3, there is little change in \( \rho_{\text{cor}} \) in the region of larger couplings \( (\mu_1 \approx 1) \). For example, when \( z \approx z_R, \rho_{\text{cor}} \approx 0.99 \) and \( \mu_1 \approx 0.89 \).

It is worth noting that the correlation coefficient defined in [7,11] can only be used for quantifying the first-order amplitude couplings. However, the amplitude coupling coefficient defined by (3) can be used for quantifying arbitrary order amplitude couplings, such as higher-order spatial chirp caused by high-order angular dispersion (naturally occurring after prisms and gratings). For example, when the second-order angular dispersion of an angular disperser is considered alone, the electric field of Gaussian ultrashort pulses with Gaussian spatial profiles at an arbitrary distance from an angular disperser can be expressed in the frequency domain as

\[ E(x, \Omega) = b_1 \exp \left( -\frac{r^2 - \Omega^2}{4} \right) \exp \left( -\frac{k_z x^2}{2z} \right) \]  
\[ \times \exp \left[ \frac{ik_z x^2}{2z} \frac{\beta(d + \alpha^2 z)}{2} \right] \]  
\[ \text{(11)} \]

where \( \beta = \partial \partial_\Omega \) is the second-order angular dispersion factor of the angular dispersion component. From (11), there exists second-order spatial chirp. The corresponding amplitude coupling coefficient can be calculated by using (3) and (11), and their properties are shown in Fig. 3.

![Fig. 3. Correlation coefficient and amplitude coupling coefficients as a function of propagation distance after a grating, \( \mu_a (i=1,2) \) are the amplitude coupling coefficients for the first-order and second-order spatial chirp respectively. Calculating parameters for \( \mu_a : \alpha = 0.27, \beta_1 = 0.29 \) fs, \( \beta_2 = 0.5 \) mm, \( d = 0.98 \) m, \( \lambda_0 = 800 \) nm, and \( T = 2.73 \) fs. Calculating parameters for \( \mu_c : \beta_1 = 0 \) fs, \( \beta_2 = 0.12 \) fs², and other parameters are the same with those of \( \mu_a \).](image_url)

As shown in Fig. 3, although the trend of the \( \mu_a \) is similar to that of \( \mu_c \), \( \mu_a \) is much smaller than \( \mu_c \) at a certain propagation distance \( z \). It means that the spatial chirp is mainly caused by the first-order angular dispersion of the grating.

IV. ANGULAR DISPERSION

In the \( k-\omega \) domain, the amplitude couplings are angular dispersion [7]. The schematic diagram of angular dispersion can be seen clearly in Fig. 1 of [9]. Similarly to (3), the amplitude coupling coefficient in the \( k-\omega \) domain can be defined by replacing \( x \) with \( k \) in (3).

Gaussian ultrashort pulses with first-order spatiotemporal couplings in the \( k-\omega \) domain can be expressed in the form [7]

\[ E(k, \Omega) < \exp \left\{ S_{\omega} k^2 + 2S_{\omega} k \Omega - S_{\omega} \Omega^2 \right\} \]  
\[ = \exp \left\{ H k^2 + 2iH k \Omega - J \Omega^2 \right\} \]  
\[ \times \exp \left\{ H (k^2 + 2L k \Omega - M \Omega^2) \right\} \]  
\[ \text{(12)} \]

In (12), \( H = S_{\omega}, I = S_{\omega}, J = S_{\omega}, K = S_{\omega}, L = S_{\omega}, M = S_{\omega} \). \( I \) and \( L \) are the coupling coefficients for amplitude couplings and phase couplings, respectively. Because \( H \) and \( J \) are related to angular divergence and bandwidth, respectively [7], \( H \) is always negative, whereas \( J \) is always positive. Similar to (5), the amplitude coupling coefficient for an ultrashort pulse described by (12) can be deduced as
\[
\mu_a = 1 - \left( \frac{I^2 + HJ}{HJ} \right)^{1/2}, \tag{13}
\]

By using the Eq. (2.133) of [3], the electric field of Gaussian pulses with Gaussian spatial profiles at an arbitrary distance from an angular disperser can be expressed as

\[
\tilde{E}(k_z,\Omega) = b_4 \exp(-\frac{\tau^2\Omega^2}{4}) \times \exp \left[ \frac{iq(d)}{2k_c\alpha^2} (k_z - k_s\beta\Omega)^2 \right] \exp \left[ -ik_s(1 - \frac{k_s^2}{2k_c^2})z \right], \tag{14}
\]

where \(b_4\) is an amplitude constant, and \(k_s\) is the spatial frequency.

It is easy to reduce (14) to the form of (12), and the corresponding amplitude coupling coefficients can be obtained by using (13), i.e.,

\[
\mu_a = 1 - \left( \frac{\alpha^2\tau^2}{\alpha^2\tau^2 + 2k_c^2z_c\beta^2} \right)^{1/2}. \tag{15}
\]

Equation (15) implies that angular dispersion caused by an angular disperser depends on the angular dispersion parameters \((\alpha, \beta_i)\), the pulse width parameter \(\tau\) and the Rayleigh length \(z_R\) intimately, but does not depend on the distances \(z\) and \(d\). It means that the magnitude of angular dispersion caused by an angular disperser keeps constant as the beam propagates in the free space. For a monochromatic light beam \((\tau \to \infty)\), equation (15) yields \(\mu_a = 0\), indicating that angular dispersion does not exist in a monochromatic light Gaussian beam. For the case of \(\beta_1 = 0\), there is \(\mu_a = 0\), meaning that angular dispersion does not exist in the absence of angular dispersers.

Finally, to measure amplitude coupling coefficient, one can use a self-referencing technique [1] or SEA TADPOLE technique [16], etc.

V. CONCLUSIONS

In conclusion, we define an amplitude coupling coefficient to quantify spatiotemporal couplings in ultrashort pulses. This amplitude coupling coefficient is dimensionless and normalized, and readily indicates the severity of amplitude couplings. With the amplitude coupling coefficient, the first-order spatial chirp and angular dispersion of Gaussian pulses with Gaussian spatial profiles are described analytically. Furthermore, we study the second-order spatial chirp caused by angular dispersion elements with the amplitude coupling coefficient. It can be shown that the amplitude coupling coefficient defined in this paper can help better understand spatiotemporal couplings and their consequences, and may be used as a benchmark to evaluate the severity of all possible amplitude couplings.

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REFERENCES