THE CONTROL OF OSCILLATORY MODES IN A BANG-BANG TRACKER

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Abstract
Phase plane methods are developed for the analysis of oscillations in pulsed systems which can be modeled as sampled-data bang-bang systems. The methodology is graphical and provides a base for the synthesis of lead compensation networks for controlling the oscillatory problem.

Introduction
The objective of this paper is to present an analysis, by means of an example, of simple lead compensation on moding in a sampled-data bang-bang system with second order plant dynamics.

A laser tracker is a system where the control of moding is of concern and will be used as an example.

A laser tracker designed to track a point which is being illuminated by a pulsed laser whose pulse repetition rate is $f$, and which employs a quadrant detector on a gimbaled platform, yields the best steady state pointing accuracy when its oscillations about the null point are constrained to a frequency of $f/2$. This oscillatory mode is the primary mode. However, since the tracker is nonlinear its response is input dependent and steady state oscillations in modes other than the primary mode can occur. This is shown using phase plane methods in the following section followed by an example where lead compensation is used to eliminate the problem. The problem as presented is constrained to tracker motion in one plane.

Using a Sampled Data Bang-Bang Tracker
Figure 1 is a block diagram of the system being considered. It is assumed that the target being tracked is located along the null axis of the sensor/platform combination; that is along the caged (boresight) optical axis thus providing an initial pointing error of zero. The nonlinearity is bang-bang and represents the laser detector and signal processing in one plane. The sample and hold representation models the fact that the information to the platform is via the reception of a pulsed laser spot. Reflected energy falls on one side of the detector thus setting a signal of predetermined amplitude whose polarity remains constant for at least one time interval $T=1/f$. The remaining blocks depict dynamics associated with the gimbaled platform.

Figure 2 is a signal flow graph representation of Figure 1. The use of this signal flow graph is documented in Reference 1 and an understanding of such is necessary for the reading of this paper.

The point in time at which the system is released is denoted as $t_0$. The value of $c(t=t_0)$ determines $m(t)$, which in this example is -1 or +1 for at least one laser pulse interval $T$. The states $v(t)$ and $c(t)$ can be computed over one laser pulse interval by using $m(t)$ and the initial conditions $v(t_0)$ and $c(t_0)$. The following uses Figure 2 for deriving the system's phase plane equation.

Application of Mason's gain formula to the signal flow graph yields,

$$C(s) = \frac{c_0 + \frac{v_0}{s}}{s(\alpha s + 1) + \frac{Km_0}{\alpha s}} + \frac{Km_0}{\alpha s}, \quad (1)$$

$$V(s) = \frac{v_0}{s + \alpha} + \frac{Km_0}{\alpha}, \quad (2)$$

where $c_0$, $v_0$, $m_0$ denote values at $t=t_0$. The inverse Laplace is

$$c(t) = c_0 + \left(1 - \frac{1 - e^{-a(t-t_0)}}{a} \right)v_0 - \left( \frac{\alpha(t-t_0)}{a} - 1 + \frac{e^{-a(t-t_0)}}{a} \right)Km_0, \quad (3)$$

$$v(t) = e^{-a(t-t_0)}v_0 + \left(1 - \frac{1 - e^{-a(t-t_0)}}{a} \right)Km_0, \quad (4)$$

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Noting at this point that

\[-\frac{1}{a}v(t) = \left(\frac{1}{a^2} + \frac{1}{a^2}e^{-(t-t_0)}}\right)K_m - \frac{1}{a}s(t-t_0)\nu_0, \tag{5}\]

equations (3) and (5) can be combined to yield

\[c(t) = L(t-t_0)K_m + \frac{c_0}{a} + \frac{1}{a}\nu_0 - \frac{1}{a}v(t). \tag{6}\]

Equation (4) is solved for \(t-t_0\),

\[t-t_0 = -\frac{1}{a}\ln\left(\frac{K_m/a - \nu(t)}{K_m/a - \nu_0}\right), \tag{7}\]

equations (6) and (7) are combined to yield

\[c(t) = c_0 - \frac{1}{a}(v(t)-\nu_0) - \frac{1}{a^2}K_m\ln\left(\frac{K_m/a - \nu(t)}{K_m/a - \nu_0}\right) \tag{8}\]

which is the system's phase plane equation.

Time, in the phase plane, can be tracked through equation (6). Since emphasis is placed on the conditions at a sampling instant and the new values of \(c\) and \(v\) after one sample interval \(T\), that is \(t=0\) to \(T\), equation (6) is written as,

\[c_{n+1} + \frac{1}{a}v_{n+1} = c_n + \frac{1}{a}\nu_n + \frac{K_m}{a}T, \tag{9}\]

where \(n\) denotes initial conditions at some sampling instant \(t=0\) and \(n+1\) denotes new values after one interval \(T\).

Clearly, if at a given sampling instant \(t=0\), \(c_0\) and \(v_0\) are known and since \(c_0\) defines the value of \(m_0\), these values are the initial conditions for the next time interval \(T\), Equation (8), with these initial conditions, defines the system's trajectory over an interval \(T\), until \(t=(n+1)T\). This point occurs in the phase plane when the trajectory intercepts the line defined by equation (9). This intercept defines new initial conditions and the process is repeated. This is illustrated through the following example.

Let conditions at the first sampling instant be \(c_0=\nu_0=0\),

\[G(s) = \frac{40}{s(s+100)} \tag{10}\]

and let the interval between laser pulses be \(T=0.1\) seconds. Equation (8), the system's phase plane equation is

\[c(t) = -0.1v(t) - 0.04\ln\left(\frac{0.4 - v(t)}{0.4}\right) \tag{11}\]

if \(m_0\) is positive and

\[c(t) = -0.1v(t) + 0.04\ln\left(\frac{0.4 + v(t)}{0.4}\right) \tag{12}\]

if \(m_0\) is negative. Also, define \(f(e)\) as being positive for \(e>0\), which sets \(m\) positive as soon as a sample is taken. Equation (11) defines the trajectory in the phase plane through the points of initial conditions; \(c_0=\nu_0=0\). The switching line is defined by equation (9). The trajectory intercepts this line and is redefined through equation (11) for a new set of initial conditions and a negative value of \(m\). The switching line intercept points are used to define the next switching line. The results of this trajectory tracing procedure is shown in Figure 3 and it is observed that the system is oscillating in a mode whose frequency is one half of the primary mode. The system is said to be double moding. The objective of the following section is to show how lead compensation can eliminate this particular problem.

The Control of Moding

The primary mode for the previous example is \(f=1/2T=5\) Hz. Initial conditions selected yielded an oscillatory mode of \(f=1/4T=2.5\) Hz. Other combinations of conditions may be found for which the system double modes. The previous example assumed the system was released at the same time a sample was taken. This is not a restriction and double moding can be obtained due to timing relationships also.

Assume that the system incorporates lead compensation as shown in Figure 4. Figure 5 is the equivalent of Figure 4 and used in the following analysis. The error signal to the relay is

\[e(t) = -nc -\nu. \tag{13}\]
The utility of this is shown by a continuation of the preceding example. Select $n$ as 10 and allow the rest of the problem parameters to remain the same. Equation (13) is drawn in the phase plane in Figure 6 and provides a boundary below which the error signal, $e$, is positive and above which it is negative. The trajectory starts from the point of initial conditions, $c_0 = v_0 = 0$, until it intercepts the switching line. Since this point of intercept is above the error signal boundary line $e$ is negative and likewise the selection of $m$ is negative in the system's phase plane equation. The trajectory is continued until the next switching line is intercepted. Since this point is below the error signal boundary line $e$ is positive and the trajectory changes rather than being continued as in Figure 3. It is now noted that the period of oscillation is $2T$ and the system is in the primary mode.

Additional examples are given in Figures 7, 8, and 9. These examples are for the case where $K = a = 1.0$ and $r$ is a unit step input. The heavy lines in Figures 7 and 8 depict a double mode situation. Figure 9 shows a triple mode of period $6T$. The dashed line is the error signal boundary line for $n = 0.7$ and the dashed trajectories are for the lead compensated cases. These examples were taken out of Reference 1.

The initial conditions for Figure 8 are such that equation (8) does not immediately apply because the term whose natural log is taken is initially negative. Equations (4) and (6) are used for obtaining the phase plane trajectory until equation (8) applies.
Conclusion

Careful examination of the examples given in this paper and others not presented here, coupled with analog computer modeling of the problem, sets a basis for the following conjecture.

Systems of the type modeled by Figure 1 for the autonomous and step input case can be constrained to primary mode oscillations by lead compensation when $n \leq a$ as defined in Figure 5.

References