A fluid action with galilean electromagnetism

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Abstract—This article represents a study of energy transport in electrolytic bubbly flows. The work models the electrodynamics of linear isotropic materials using galilean electromagnetism which is reformulated in a modern geometric language. The paper expresses an action functional of galilean fluid electrodynamics and derives an energy flux based on Hamilton’s principle. The article further provides an introduction to contemporary galilean relativity.

Index Terms—bubbly flow; electrolyte; galilean electromagnetism; milne spacetime; Hamilton’s principle

I. OPENING

From Hamilton’s principle, the dynamics of a system are derivable from an action functional which can also be used to reformulate classical physics into more modern approaches [1]. An action furthermore provides insight into a system’s behaviors, notably the stability of physical observables (such as in [2]) and conservation laws via Noether’s theorem. Additionally, the lagrangian of the action is expressed in terms of field quantities and is thus invariant with regards to generalized coordinates. Generally speaking, there is no lack of literature on various action functionals of classical and relativistic fluids, i.e. [2]–[13].

The lagrangian, being a spacetime invariant, lends itself naturally to a formulation in terms of geometric quantities, specifically i.e. in terms of tensors and differential forms. However, covariant formalism, ubiquitous in relativistic physics, is only atypically brought to bear for problems of galilean-newtonian physics. In so doing, the implications of the geometric structure underlying the physics is lost and, with it, mathematical insights into the dynamics of the system. One such insight is the invariance of newtonian mechanics under galilean transformations. But, like einsteinian physics, newtonian mechanics is in fact consistent with a geometric formulation [11], [14]. Thus, relativistic insights are retained without needing to stir up controversies of einsteinian physics such as how to interpret thermodynamic observables (for examples see [15] and [16]). By way of an example, consider a barotropic fluid of velocity \( \vec{v} \) and vorticity \( \vec{\omega} := \text{curl} \; \vec{v} \), dual on flat 3-dimensional euclidean space to a differential 2-form \( \omega_{ab} \) having components

\[
2\omega_{ij} := (dv)_{ij} := 2\partial_i v_j := \partial_i v_j - \partial_j v_i
\]

(1)

with \( i, j \in \{1, 2, 3\}, d \) the exterior derivative and \( \partial_i \) the partial derivative with respect to the \( i \)th coordinate. The indices on the vorticity 2-form, and all indices in this article from the beginning of the roman alphabet, serve only to indicate the structure of spacetime tensors. Now, a traditionally formulated but involved derivation shows that the helicity, i.e. the inner product of the fluid velocity and vorticity, is conserved in the sense that its time derivative is equivalent to a vector’s divergence [17]. That is \( \partial_t (\vec{v} \cdot \vec{\omega}) = \text{div} \; \Psi \) for a specific vector \( \Psi \). However, helicity conservation is expressible almost trivially in terms of a spacetime current 3-form [7] which is naturally derivable from an action.

Unsurprisingly, fluid electrodynamics too has benefitted from a variational formulation. For example, a functional approach can show that perfect magnetohydrodynamics obeys a kelvin-like circulation conservation law that includes the magnetic field [9]. However, unlike newtonian mechanics, classical electrodynamics is not galilean invariant because of the well-known fact that the full set of Maxwell’s equations is consistent with special but not galilean relativity. Thus, the underlying functional of fluid electrodynamics is only approximately galilean invariant [12]. On the other hand, Bellac and Lévy-Leblond [18] show that the magnetostatic and electrostatic limits of electrodynamics are indeed galilean invariant. Obviously though, these limits do not reproduce the full gamut of electrodynamic phenomena. Bellac and Lévy-Leblond do however indicate that the vacuum and material maxwell equations, i.e. those with and without sources, are themselves in fact galilean invariant. That is, in the galilean approach to electromagnetism, the vacuum and material fields are distinct but coupled through constitutive relations in an \textit{ab initio} preferred frame. In this way, instead of modeling materials as done in classical electrodynamics, materials are built into the galilean invariant theory out of necessity. As shown below, the lorentz force of this tack approximates that of classical electrodynamics with linear isotropic materials for all observers with nearly the same 4-velocity as the preferred frame.

Ultimately, this article represents an effort to, rather generally, understand bubbly flows in seawater. As such, the article expresses a means to specialize a galilean fluid electrodynamics action to poorly conductive flows. Multiphase flows, or more specifically energy fluxes between different regions, are studied by restricting the action functional to holey spacetimes. Specifically, the action’s variation over its boundaries are related to the flux of physical observables and, in this context, energy transport arises naturally.

The platforms upon which the energy flux is derived are galilean relativity and galilean electrodynamics. These topics are discussed in sections II and III respectively. The discussion of galilean relativity is in line with the formulation in [11]. The approach in section III is a revamp of traditional galilean
electromagnetism [18] into a modern, and more transparent, language. Then, in section IV, a fluid action is given that models weakly conductive flows using this galilean invariant formalism. Energy flux is discussed in this context.

II. GALILEAN RELATIVITY

Galilean relativity, like special and general relativity, is based on the machian principle that equations of motion should be the same for all inertial observers. Unlike einsteinian relativity, galilean relativity presupposes a universal time coordinate \( x^0 := ct \) for some conversion factor \( c \) that parametrizes space. Note that this article employs geometrized units where the speed of light \( c \) is unity so that \( x^0 = t \). Galilean-newtonian physics is then in harmony with the picture that spacetime is foliated by hypersurfaces of flat space for which all events occur for all observers simultaneously. This is contrary to einsteinian mechanics. Cartesian spacetime coordinates of different observers, depicted by different time axes, are shown for galilean and special relativistic observers in figure 1. Hypersurfaces of simultaneity are depicted as lines of constant time, \( i.e. \) leaves of a spacetime foliation. Coordinates compatible with galilean relativity are setup on spacetime in the following manner. Any observer, with \( \mu \)-dimensional cartesian coordinates \( x^\mu \) and therefore spacetime leaves of a spacetime foliation

\[
\text{Fig. 1. Cartesian coordinates for different observers in galilean (top) and special (bottom) relativity. In galilean relativity, all observers share the same hypersurfaces of simultaneity. In special relativity, lorentz transformations rotate the time and space coordinates into one another.}
\]

Separately note that matter fields, such as the observer, are nor-

malized in that \( u^a t_a = 1 \) where the temporal 1-form \( t_a := \partial_a t \) is the 4-dimensional gradient of the time parameter. There is no ambiguity in this last condition because the temporal 1-form and the zeroth component of the 4-velocity are physical in the sense that they are reference frame independent.

There is no \textit{ab initio} geometric reason, as Milne realized [19], to prefer one set of inertial observers over any other in a general spacetime compatible with galilean-newtonian physics. That is, a manifold with a universal time parameter alone does not inspire a preferred set of “non-accelerating” observers. Thus, galilean-newtonian physics is invariant with respect to milne-type coordinate transformations \( x^\mu \rightarrow (t, x^i - f^i) \) for the time-dependent shift parameters \( f^i := f(t)^i \). Under milne coordinate changes, vectors transform via the pushforward

\[
\Lambda^\mu_\nu = \delta^\mu_\nu - \hat{f}^\mu t_\nu
\]

where \( \delta^\mu_\nu \) is the kronecker delta for the shift-velocity components \( \hat{f}^\mu := \partial_t f^\mu \) where \( f^\mu := (0, f^i) \). Similarly, covectors transform via its inverse. These transformations are more general than boosts which occur only when the shift parameters are time-linear. That is, for every distinct set of

milne coordinates on spacetime there is a 3-parameter family of boosts that relate inertial observers.

Distances are measured for a given observer using the metric

\[
ds^2 := h_{\mu\nu} \partial_\mu x^\mu \cdot \partial_\nu x^\nu = h_{ij} \partial_i x^i \cdot \partial_j x^j
\]

where the component matrix \( h_{ij} \) of the spatial metric is simply the identity. (The dot here is used only to isolate the action of the derivative.) Repeated concrete indices are summed according to einstein notation. The spacetime metric tensor \( h_{ab} \) is orthogonal to the observer’s aether field \( a^a \) in that \( h_{ab} a^b \equiv 0 \) and thus its components \( h_{\mu\nu} \) form a degenerate rank-3 matrix. As such, \( h_{ab} \) is a riemannian metric of each spatial slice but not technically a spacetime metric. Another noteworthy feature is that there is one such metric tensor for
every distinct aether field. Similarly, the metric is not galilean invariant because it transforms as

\[ h_{ab} \mapsto h_{ab} + 2\dot{f}(a)t_b + \dot{f}^2t_at_b \]

(5)

where \( \dot{f}_a := h_{ab}\dot{f}^b \) and \( \dot{f}^2 := \dot{f}_a\dot{f}^a \) and the parentheses on the indices denote symmetrization, i.e.

\[ 2\dot{f}(a)t_b := \dot{f}_a t_b + \dot{f}_b t_a. \]

(6)

Accordingly, the degenerate metric of galilean-newtonian relativity is sharply contrasted with the non-degenerate lorentzian metric of einsteinian relativity which is unique and invariant to special relativistic, i.e. lorentz, boosts. (See section III for more details.) On the other hand, the metric “inverse” \( h^{ab} \), or contravariant projector, defined by promoting the tensor to spacetime, \( i.e. h^{\mu\nu} := h^a\partial_\nu x^a \cdot \partial_\mu x^a \), is in fact milne invariant. Additionally, the projector is normal to the metric 1-form, \( i.e. t_a h^{ab} \equiv 0 \). Thus, the tensor \( h^{ab} \) projects onto hypersurfaces that are level sets of the time parameter. Since the metric and its inverse are degenerate, they cannot be used to uniquely identify general spacetime tensors with elements in their dual space. For example, \( \Psi^a \neq h^{ab}h_{bc}\Psi^c \) for all fields \( \Psi^a \). Despite this hitch, this article readily uses the metric to lower contravariant indices such as in (5).

The structures discussed so far are sufficient to define milne and galilean geometries. A milne geometry requires the triple \((R^4, t, h^{ab})\), \(i.e.\) a manifold taken here as merely the 4-dimensional euclidean topology \( R^4 \), and an inverse metric \( h^{ab} \) that resides in the level sets of the universal time parameter. But, strictly speaking, under the assumption that spacetime is flat, there is no actual need to assume \( h^{ab} \) exists. Alternatively, galilean geometries \((R^4, t, h^{ab}, a^a)\) have an aether field to define a preferred set of inertial observers. An important note is that measurements such as kinetic energy require a metric or any frame related to it by simply a boost, the christoffel symbol vanishes. Clearly this derivative is curvature-free, \(i.e.\)

\[ 2\nabla_a\nabla_b\Psi^c \equiv 0, \]

(11)

since it reduces to the partial derivative in at least one coordinate system. This derivative also annihilates the temporal 1-form and the spatial projector, \(i.e.\) \( \nabla_a t_b \equiv 0 \) and \( \nabla_a h^{bc} \equiv 0 \). Furthermore, it annihilates the aether field that defines its natural metric. This is seen by considering that the aether field in its cartesian coordinates has the components \( a^\mu = (1, 0, 0, 0) \) and is thus annihilated by that coordinate system’s partial derivative. Since every metric represents a particular gauge choice, the connection too is gauge dependent. However, once the gauge is set, the derivative acts tensorially and thus may be used to covariantly express physical laws.

III. GALILEAN ELECTRODYNAMICS

Electromagnetism in special relativity is formulated thusly [20]. The geometry is \((R^4, \tilde{\eta}_{ab})\) where the components of the minkowski metric in cartesian coordinates are

\[ \tilde{\eta}_{\mu\nu} := \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

(12)

The minkowski metric, unlike any one of the many degenerate metrics of milne geometries, has a legitimate inverse \( \tilde{\eta}^{ab} \) and so can be used to freely raise and lower tensor indices. This technique is applied liberally below. A lorentz boost, in the \(x^1\) direction with speed fraction \( \beta := v/c = v \) for some speed \(v\), transforms the coordinates of one inertial frame into another and is given by

\[ x^\mu \mapsto \left( \gamma(x^0 - \beta x^1), \gamma(-\beta x^0 + x^1), x^2, x^3 \right) \]

(13)

where \( \gamma := (1 - \beta^2)^{-1/2} \) is the lorentz factor. The bottom of figure 1 shows coordinates under such a boost transformation. The differential \( \tilde{\Lambda}_\mu^\nu \) of this lorentz transformation is

\[ \tilde{\Lambda}_\mu^\nu = \begin{bmatrix} \gamma & -\gamma\beta & 0 & 0 \\ -\gamma\beta & \gamma & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}. \]

(14)

Importantly, the minkowski metric is invariant under such transformations. This is in stark contrast to the metric of galilean relativity which is not invariant under galilean boosts. The faraday 2-form \( F_{ab} \) succinctly encodes the relativistic lorentz force

\[ F_{\mu
u} = q\tilde{\eta}^{ab}F_{bc}\dot{a}^c \]

(15)
experienced by a charge \( q \) with 4-velocity \( \dot{u}^a \) normalized such that \( \dot{u}^a \dot{u}^b \eta_{ab} = -1 \). It is constructed from the electric and magnetic fields, \( i.e. \)

\[
\vec{E}_a := \vec{F}_{ab} \dot{u}^b \quad \text{and} \quad \vec{B}_a := -\frac{1}{2} \epsilon_{abc} \vec{F}_{cd} \dot{u}^b
\]

(16)
in a frame generated by \( \dot{w}^a \). The magnetic field is isolated using the invariant spacetime volume \( \epsilon_{abcd} = \sqrt{\det(\eta)} \epsilon_{abcd} \) where \( \epsilon_{abcd} \) is the right-handed coordinate volume 4-form. These fields are purely spatial in this frame in that the right hand sides vanish identically when contracted with another copy of \( \dot{w}^a \). Expressly, the faraday tensor is then

\[
\vec{F}_{ab} = 2 \dot{u}_a \vec{E}_b + \epsilon_{ab} \dot{u}_d \vec{B}_d.
\]

(17)

Here a moment is taken to scribble some standard notation and relationships \[21\]. The wedge product of the \( p \)-form \( \tilde{\rho} \) and \( q \)-form \( \tilde{\eta} \) is defined as

\[
\tilde{\rho} \wedge \tilde{\eta} := (p + q)! \tilde{\rho}_{[A} \tilde{\eta}_{B]} \]

(18)

where \( A \) and \( B \) are multi-indices, here representing \( p \) and \( q \) abstract indices respectively. The hodge dual on minkowski space maps \( p \)-forms to \( (4 - p) \)-forms as

\[
\ast \tilde{\rho} := \frac{1}{(4 - p)!} \epsilon_A^{\\abct} \tilde{\rho}_A
\]

(19)

The exterior derivative \( \ast \) is defined to act on differential forms as \( d \tilde{\rho} := (p + 1) \partial_{[A} \tilde{\rho}_{B]} \). Note that the exterior derivative is equivalently expressible in terms of any torsion-free connection, \( e.g. \) \( d \tilde{\rho} = (p + 1) \nabla_{[A} \tilde{\rho}_{B]} \). Accordingly the exterior derivative acts tensorially. An additional property of the exterior derivative is that it commutes with coordinate transformations in the following sense. Consider a coordinate transformation between manifolds \( M \) and \( N \) given by the diffeomorphism \( \Psi : M \rightarrow N \). Take the \( p \)-forms and the exterior derivatives on these manifolds as \( p_M, d_M \) and \( p_N, d_N \). The pullback \( \Psi^\ast \), which is induced by the diffeomorphism’s differential, maps differential forms on \( N \) to \( M \), \( i.e. \) \( p_M = \Psi^\ast p_N \). The exterior derivative is natural with respect to the pullback in that \( d_M(\Psi^\ast p_N) = \Psi^\ast (d_N p_N) \).

Back to the point at hand, Maxwell’s equations are expressed readily using the faraday 2-from. The source-free maxwell equations—these equations are \( d \vec{B} = 0 \) and \( \text{curl} \ \vec{E} + \partial_t \vec{B} = 0 \) in standard notation—are equivalent to the annihilation of the faraday 2-form by the exterior derivative, that is

\[
d \ast \vec{F} = 0.
\]

(20)

On the other hand, the sourced maxwell equations—expressed in terms of the volume current \( \tilde{j} \), charge density \( \rho \) and the permittivity and permeability of free-space \( \varepsilon_0 \) and \( \mu_0 \) as \( \text{curl} \ \vec{B} = \partial_t \vec{E} + \mu_0 \tilde{j} \) and \( \text{div} \ \vec{E} = \rho / \varepsilon_0 \) in the standard notation—are provided by

\[
d \ast \vec{F} = -3 \mu_0 \ast \tilde{j}
\]

(21)

where \( \tilde{j}_a \) is the current 1-from. For linear isotropic media these equations are \( d \vec{D} = \rho_t \) and \( \text{curl} \ \vec{H} = \partial_t \vec{D} + \tilde{j}_t \) in the standard notation. Finally, Maxwell’s equations in (20) and (21) necessarily hold in all inertial frames because the exterior derivative is natural with respect to coordinate transformations.

Now, any electromagnetic theory invariant under galilean boosts must be distinct from classical electrodynamics. Simply put, the electrodynamic formulation above relies on the non-degenerate and lorentz boost invariant minkowski metric. And even though any frame in galilean relativity can be equipped with the minkowski metric, that metric is not galilean invariant. One way to circumvent this issue is to rewrite Maxwell’s equations as formulae of distinct variables. Put another way, the vacuum and sourced field equations separately are in fact milne invariant. Galilean electromagnetism is thus constructed by dressing the field variables in two separate guises, that of the electric \( E^a \) and magnetic \( B^a \) fields and the material fields \( D^a \) and \( H^a \). This theory approximates classical electrodynamics in a well-defined manner described below.

Like standard electromagnetism, the galilean electric and magnetic fields are not tensors themselves but rather represent different components of the same tensor. Ergo, they are generally different for every observer. For an observer with 4-velocity \( w^a \) and natural metric \( h_{ab} \) the electric and magnetic fields form the components of the galilean faraday 2-form

\[
F_{ab} = 2E_{[a}h_{b]} + \epsilon_{abc}w^c = 2E_{[a}h_{b]} + \epsilon_{abc}B^c.
\]

(22)

Here, \( \epsilon_{abcd} = \sqrt{\det(h)} \epsilon_{abcd} \) is the spacetime volume form whereas \( \epsilon_{abc} := w^d \epsilon_{dabc} \) is the volume form in the observer’s gauge. Although the first two terms of the galilean faraday tensor appear to be dependent on a specific metric, taken together they actually are not. Consider that the metric transforms as in \( 5 \) and so generally has non-zero temporal and spatiotemporal components. These terms are however mapped to zero by either the electric field itself, which is presumed to be spatial, or by antisymmetrization with the temporal 1-form. In a frame for which \( w^a \) has 3-velocity \( v^a \), the electric and magnetic fields are

\[
E'_a := E_a - \epsilon'_{abc}v^b B^c \quad \text{and} \quad B'^a := B^a
\]

(23)

where \( \epsilon'_{abc} \) is the volume 3-form in the new frame. In form notation the galilean faraday tensor is

\[
F = E \wedge t + \ast_g (w \wedge B).
\]

(24)

The expression here extends the wedge product in the obvious fashion to also act on vectors (and not just differential forms). The hodge-like duality operator \( \ast_g \) uses the spacetime volume form \( \epsilon_{abcd} \) to map tensors of type \((p, 0)\) to \((4 - p)\)-forms in similar fashion to \((19)\). By the same line of reasoning as held in the relativistic case, the vacuum galilean maxwell equations

\[
dF = 0.
\]

(25)

are frame independent because the exterior derivative commutes with the pullback.

Similarly, the galilean material fields are components of the 2-form

\[
G = -H \wedge t + \ast_g (w \wedge D).
\]

(26)
Again, in a frame for which the 4-velocity \( u^a \) has spatial components \( v^i \), these field variables transform similarly as

\[ H'_a := H_a + \epsilon_{abc}v^bD^c \quad \text{and} \quad D'^a := D^a. \quad (27) \]

The material galilean maxwell equations are given by

\[ dG = -3! \star_g j_t \quad (28) \]

where \( j^a_t \) is the free charge 4-current. Clearly, these equations too are milne frame independent.

The price to be paid for the milne invariance of the galilean maxwell equations is that they now constitute two independent sets of field equations. What is worse is that the vacuum equations are entirely unrelated to the charge and current distributions. To have any hope that these equations can emulate standard electrodynamics, the fields must be related. Rather generally however, how the fields should relate to one another depends on the system in question, i.e. different constitutive relations model different media. For example,

\[ \varepsilon \tilde{E}^a = \tilde{D}^a \quad \text{and} \quad \tilde{B}^a = \mu \tilde{H}^a \quad (29) \]

for linear isotropic media in standard electrodynamics. Similar equations cannot hold in general for galilean electrodynamics because the field variables abide by different transformation rules, namely by (23) and (27). However these relations are enforceable for at least one reference frame. The aether field that defines the preferred set of inertial observers in galilean geometries provides exactly this necessary structure. The aether vacuum and material fields are then

\[ E^a := h^{ab}F_{bc}a^c, \quad \tilde{B}^a := \frac{1}{2}\epsilon^{abcd}F_{cd}b^b, \]

\[ D^a := \frac{1}{2}\epsilon^{abcd}G_{cd}l_b \quad \text{and} \quad \tilde{H}^a := -h^{ab}G_{bc}a^c \quad (30) \]

where \( \epsilon^{abcd} \) is the fully antisymmetric inverse of the spacetime volume 4-form, normalized so that \( \epsilon_{abcd}\epsilon^{abcd} = -4! \). Thus, linear isotropic media in galilean electrodynamics are materials for which the vacuum and displacement fields are proportional in the aether rest frame, i.e.

\[ \varepsilon \tilde{E}^a = \tilde{D}^a \quad \text{and} \quad \tilde{B}^a = \mu \tilde{H}^a. \quad (31) \]

More to the point, traditional galilean electrodynamics is a theory of linear isotropic media.

The goal now is to show that galilean electrodynamics approximates its relativistic counterpart. More specifically the goal is to show that the galilean lorentz force

\[ f^a_L := qh^{ab}F_{bc}u^c \quad (32) \]

on the test 4-current \( qu^a \) approximates the lorentz force (15) on the relativistic 4-current \( q\tilde{u}^a = q\gamma_au^a \). The comparison can be made because there is no problem with equating the coordinates of any specific inertial frame in galilean and special relativity even though coordinates that are galilean transforms are not lorentz transforms (this is depicted in figure 1). That is, an inertial observer on minkowski space \( (R^4, h_{ab}) \) with cartesian coordinates \( x^\mu \) can establish the aether field \( a^a \) as in (2). Moreover, this aether field provides exactly the structure necessary to equip that geometry with a galilean structure, i.e. \( (R^4, x^\mu, h^{ab}, a^a) \). Furthermore, presume that spacetime has a 4-current \( j^a_0 = \tilde{j}^a_0 + j^a_0 \) composed of bound and free parts. The free current may be taken as the source of the galilean material maxwell equations, i.e. \( j^a_0 = \tilde{j}^a_0 \). Accordingly, the field equations of these two theories coincide in this (aetherial) inertial frame and so the components of the faraday 2-forms \( F_{ab} \) and \( \tilde{F}_{ab} \) are equal.

Now, consider the special relativistic (15) and galilean (32) lorentz forces on the test charge \( q \) with arbitrary 4-velocity \( \tilde{u}^a = \gamma_qu^a \). The test 4-currents \( q\tilde{u}^a \) and \( qu^a \) are necessarily distinct due to length contraction caused by lorentz boosts. The spatial components of these forces are proportional by the lorentz factor, that is

\[ \tilde{F}^a_L = \gamma_qf^a_L. \quad (33) \]

The lorentz factor is approximately \( \gamma_q \approx 1 + \beta_q^2 \) for small test charge speed fractions. And so, the spatial components of these forces agree at first-order in \( \beta_q \). The relativistic lorentz force does have a temporal component \( f^a_0 = \gamma_qF^a_{\mu}\beta_q^\mu \) however where \( \beta_q^\mu \) is the velocity fraction of the charge. This phenomenon is purely relativistic and has no galilean analog. That said, it only manifests at first-order in \( \beta_q^2 \). Now consider transforming the force to another frame using either the galilean (3) or lorentz (14) pushforwards. In the galilean scheme, the force is frame independent even though the electric field is not. In the other scheme, for a boost speed \( \beta_t \) in the \( x^4 \) direction and lorentz factor \( \gamma_\Lambda \), the force transforms as

\[ \tilde{F}^a_L \rightarrow \left( \gamma_\Lambda(f^a_L - \beta_\Lambda\tilde{F}^a_L), \gamma_\Lambda(-\beta_\Lambda^1f^a_L + \tilde{F}^a_L), \tilde{F}^a_L, f^a_L \right). \quad (34) \]

Accordingly, the force in the new \( x^4 \) direction is

\[ \Lambda^1_{\mu}f^a_\mu = \gamma_\Lambda(f^a_L - \beta_\Lambda\tilde{F}^a_L) = \gamma_\Lambda(\gamma_qf^a_L - \gamma_q\beta_\Lambda\tilde{F}^a_L) = f^a_L - \beta_\Lambda\tilde{F}^a_L + O(\beta_\Lambda^2) + O(\beta_q^2) \quad (35) \]

for some terms that are second-order in \( \beta_\Lambda \) and \( \beta_q \) while the force in the \( x^2 \) and \( x^3 \) directions remain unchanged. That is, the spatial components of the two lorentz forces are equal up to first-order in \( \beta_\Lambda^2 \) and second-order in the frame \( \beta_\Lambda \) and charge \( \beta_q \) boost parameters.

IV. GALILEAN FLUID DYNAMICS

By Hamilton’s principle, a physical system of field variables \( \Psi \) behaves as the minimizer of the action functional

\[ S[\Psi] := \int_M \epsilon L(\Psi) \quad (36) \]

where \( L \) is the lagrangian density and \( \epsilon \) the spacetime volume form [1]. The lagrangian of a discrete unconstrained system being merely the difference between the system’s kinetic and potential energies. In this field theoretic approach, fluids are continua of infinitesimal parcels, each built of a very large number of particles. Traditionally, action formulations accurately model only certain restricted systems, isentropic flows for instance. However, the literature is not devoid of efforts to
go beyond tradition in certain contexts [13]. The goal here is
to extract an energy transport equation for multiphase flows
from a general lagrangian for galilean fluid electrodynamics.

Just as fluid dynamics is interpreted in either lagrange
or euler variables, so too is its action. Many differences of
these two approaches are largely philosophical since they are
equivalent in the non-dissipative limit [8]. The lagrangian
perspective has now developed into the so-called convective
approach [22]. The focus here however is on the eulerian
framework. In the eulerian action, system constraints give rise
to potential flows. For example, gradient, or vorticity-free,
flows arise by imposing only conservation of mass, i.e. that
the 4-momentum is divergence-free
\[ \nabla_a p^a \equiv 0. \]  (37)

On the other hand, rotational flows result by supposing that a
scalar field \( \phi \) is conserved along the worldlines of each fluid
parcel [23]. Specifically, this “coordinate” constraint is
\[ \nabla_a \phi := p^a \nabla_a \phi \equiv 0. \]  (38)

A lagrangian of euler variables admitting both rotational
and irrotational flows may be constructed from kinetic energy
and constraints. Clearly, the kinetic energy depends on the
metric and in this way the lagrangian, though invariant, is
gauge dependent. This work employs the aether specific metric
and its levi-civita connection provided by a galilean geometry.
And so the fluid lagrangian is simply
\[ L_B := \frac{p^2}{2\rho} + \psi \nabla_a \phi - \chi \nabla_a p^a \]  (39)
where \( \psi \) and \( \chi \) are lagrange multipliers which enforce con-
straints (38) and (37) respectively. The units of both \( \chi \) and the
product \( \psi \phi \) are the same as kinematic viscosity, i.e. area per
time.

To find the minimizer, the action is perturbed using the
variational derivative \( \delta \) which assumes that the field variables
are held constant on the boundary \( \partial M \) of \( M \). Then the euler-
lagrangian equations are extracted and solved. Varying the fluid
action produces the perturbation
\[ \delta S_B = \int_M \epsilon \left[ p^a (u_a + \psi \nabla_a \phi + \nabla_a \chi) + \delta \psi \cdot \nabla_p \phi 
- \delta \phi \cdot \nabla_a (p^a \psi) - \chi \delta \nabla_a p^a \right] 
+ \int_{\partial M} \epsilon^3 \, n_a (\psi \nabla_a \phi - \chi \delta p^a) \]  (40)
where \( \epsilon^3 \) and \( n_a \) are respectively the volume form and unit
normal covector field of \( \partial M \) in \( M \) so that \( \epsilon = n \wedge \epsilon^3 \). The boundary term arises from Stokes’ theorem which implies that
\[ \int_M \epsilon \nabla \Psi a = \int_{\partial M} \epsilon^3 \, n_a \Psi a \]  (41)
for any continuously differentiable vector field \( \Psi a \). Therefore,
the euler-lagrangian equations are
\[ u_a + \psi \nabla_a \phi + \nabla_a \chi = u_a + \psi \partial_a \phi + \partial_a \chi = 0 \]  (42)
and
\[ \nabla_a (p^a \psi) \equiv 0 \]  (43)
in addition to the conservation of mass (37) and the coordinate
constraint (38) equations. Equation (43) implies conservation
of \( \psi \) along the flow lines because the momentum is divergence-
free. Projecting (42) using the inverse metric is exactly the
clebsch decomposition of a vector in euclidean space. As
such, the fluid may thus exhibit both rotational and irrotational
flows. The rotational \( u^r_a \) and rotational \( u^i_a \) parts of the flow
respectively satisfy the equations
\[ -\nabla_i (u^r_j) = \partial_i \psi \cdot (\partial_j \phi) \]  (44)
and
\[ -\nabla_i (u^i_j) = \nabla_i (\psi \cdot \nabla_j) \phi + \psi \nabla_i \nabla_j \phi + \nabla_i \nabla_j \chi. \]  (45)
Clearly, the vorticity here arises solely because of the coordinate
constraint.

The galilean maxwell equations can be incorporated into
the action functional using the constrained lagrangian density
\[ L_{em} := \frac{\epsilon_{abcd}}{2} \left( \nabla_a A_b \cdot F_{cd} + \nabla_a C_b \cdot G_{cd} \right) + C_a j^a 
+ \zeta_a \left( \frac{1}{2} \epsilon_{abcd} G_{cd} j^b - \epsilon h^{ab} F_{bc} \alpha^c \right) 
+ \xi_a \left( \mu h^{ab} G_{bc} \alpha^c + \frac{1}{2} \epsilon^{abcd} C_{cd} j^b \right) \]  (46)
for some fields \( A_a, \ C_a \). The lagrange multipliers \( \zeta_a \) and \( \xi_a \)
force the galilean electrodynamics constitutive relations (31).
Recall that these equations say that in the rest frame of the
aether the electric and magnetic fields are proportional to,
respectively, the electric and magnetic displacement fields. On
the other hand note that since the fields \( A_a \) and \( C_a \) are not
simple multipliers, these terms alter the value of the action.
The euler-lagrangian equations associated with their variations
are, respectively, the vacuum (25) and material (28) galilean
maxwell equations. The boundary terms of the lagrangian
variation are
\[ \delta S_{em}^{\partial M} := \frac{1}{2} \int_{\partial M} \epsilon^3 \, n_a \epsilon_{abcd} (F_{bc} \delta A_d + G_{bc} \delta C_d). \]  (47)

This formalism can model electrolytic flow as follows. The
prototypical method is to couple fluid flow with the free
current through the lorentz force using Ohm’s law. That is, using
\[ j^a = \sigma h^{ab} F_{ab} \] for a fluid with conductivity \( \sigma \). In the case of
an electrically neutral fluid Ohm’s law is \( j^a = \sigma h^{ab} F_{bc} \alpha^c \)
because then the free charge vanishes identically. Another
assumption commonly employed is that the magnetic inductive
effects are not strong enough to give rise to a significant
current and thus the free 3-current is generated by the electric
field only. However, since the electric field is gauge dependent
but the 3-current is not, this statement is ambiguous. However,
this approximation can still be made by specifying a particular
electric field, for instance the aetherial field \( E^a \). In this case
the current is \( j^a = \sigma h^{ab} F_{bc} \alpha^c \).

The action formulation above is applied to multiphase flows
by appropriately tailoring the domain. For example, bubbly
flows on spacetime are incorporated using the dynamic swiss cheese manifold \( \circ \) depicted in figure 2. The manifold is constructed as follows. Begin with a galilean geometry over the compact 4-dimensional set \( M \sim \Gamma \times N \subset \subset R \times R^3 \) for the temporal interval \( \Gamma = [t_1, t_2] \). Consider also the dynamic-ball topology \( B \sim \Gamma \times B^3 \) where \( B^3 \) is an open 3-ball. (A closed 3-ball in the common vernacular is simply a solid sphere and, technically, has a 2-sphere \( S^2 \) boundary.) Now, remove \( n \) dynamic-balls from \( M \) such that (i) their boundaries \( \partial B \sim \Gamma \times S^2 \) are embedded smoothly throughout \( M \) and in \( \{t_1, t_2\} \times N \) at the boundaries \( \{t_1, t_2\} \times S^2 \), (ii) their boundaries do not intersect \( \Gamma \times \partial N \) or each other and (iii) no ball is enclosed in another. Thus, what is left is the dynamic swiss cheese topology \( \circ \sim M/(x^3B) \) where each spatial slice \( \Sigma \sim N/(x^3B^3) \) has boundary \( \partial \Sigma = \partial N \cup^\circ S^2 \). The spatial and temporal boundaries of \( \circ \) are then the tubes \( \Gamma \times \partial \Sigma \) and slices \( \partial \Gamma \times \Sigma \) respectively.

In addition to reproducing a system’s governing equations, the functional formulation motivates an energy equation in the following manner. The action’s value comes from the kinetic energy and electromagnetic fields. Consider now any linear variation \( \Delta \) of the fluid (39) and electrodynamic (46) lagrangian densities. The boundary terms for their variations are given in (40) and (47). So, on shell, i.e. when the euler-lagrange equations hold, these boundary terms represent the change in the value of the action and thus an energy flux, called here the canonical work. For example, consider the spatial boundary tubes \( \Gamma \times \partial \Sigma \) where each dynamic 2-sphere \( \partial B \) represents a bubble’s 3-dimensional horizon. The total energy volume flux through the horizon is

\[
\int_\Gamma \frac{dA}{\partial \Sigma} r_a \left[ \psi p^a \Delta \phi - \chi \Delta p^a \right]
+ \frac{e_{abcd}}{2} \left( F_{bd} \Delta A_d + G_{bd} \Delta C_d \right). \tag{48}
\]

In this case, the boundary normal is simply the radial direction \( r_a \) normal to each bubble \( S^2 \) and the boundary \( \partial M \). Taking the variation along the flow of time, the canonical work at any moment for a given bubble is

\[
W := \int_{S^2} dA \ r_a \left[ \psi p^a \partial_t \phi - \chi \partial_t p^a \right]
+ \frac{e_{abcd}}{2} \left( F_{bd} \partial_t A_d + G_{bd} \partial_t C_d \right). \tag{49}
\]

But of course there is no need to restrict to temporal variations. For example, by reintroducing \( \Delta \) for \( \partial_t \), this expression holds equally well for bubble deformations within a given spatial slice.

V. CLOSING

This article reformulates galilean electromagnetism using modern differential geometry and models electrolytic flows. Employing galilean electromagnetism allows for the exploration of fluid electrodynamics without carrying around cumbersome effects such as length contraction and time dilation. Moreover, the geometric insights of a covariant theory are maintained while escaping the subtleties of, say, thermodynamics in special relativity. Finally, this work develops an expression for the canonical work of a conductive fluid using Hamilton’s principle.

REFERENCES