THE EFFECT OF NONLINEAR WAVE INTERACTIONS ON AN OFFSHORE STRUCTURE

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ABSTRACT

The effect of nonlinear wave interactions on a fixed offshore structure is examined analytically. Based on Morison's equation, the dynamic response of a single degree of freedom structure is investigated using the linear (first-order waves) and nonlinear wave interactions (second- and third-order waves) generated by two deepwater wave trains. By comparing the results, it is found that the third-order waves (resonant interactions) can significantly affect structural response.

1. INTRODUCTION

Most models for calculating wave force effects on offshore structures treat the waves as a superposition of linear wave components. Examples include the works of Arockiasamy, et al. [1], Barik and Paramasivam [2], Dao and Penzien [3], Leonard, et al. [4], and Tiagh and Hudspeth [5]. For a real sea state, the superposition of linear wave components cannot satisfy the required degree of accuracy; hence, the features of nonlinear wave interactions have to be considered.

In small amplitude, first-order wave theory, two or more simple wave trains are propagated independently and without mutual interaction. In the second-order wave theory, the interactions, including self-interactions and cross-interactions, produce only a small modification to the wave motion which remains bounded in time. Phillips [6] found that three primary waves can transfer energy to a fourth wave in such a way that the amplitude of the fourth wave increases linearly with time. Since these interactions occur in the oceans, they may produce a considerable modification in the ocean wave spectra. One particular case of Phillips' results was found by Longuet-Higgins [7] when two of the three primary waves have the same wave number.

The effect of these nonlinear wave interactions on the dynamic response of a fixed offshore structure is examined analytically in this paper. For simplicity, a single degree of freedom structure with negligible wave scattering is considered. Also, the wave loads are determined by applying Morison's equation which includes both drag and inertial force components.

In the next section, the theory of nonlinear wave interactions will be described. The structural response with and without the presence of the nonlinear wave interaction effects will be derived in section 3. The results will be shown in section 4.

2. THEORY OF NONLINEAR WAVE INTERACTIONS

The velocity potential of the nonlinear wave interaction theory has been developed by Longuet-Higgins. For the wave force consideration, the authors have extended this theory to wave kinematics, i.e., wave velocity and acceleration. The theory is given below.

First-Order Waves

Since the two wave trains considered are in deep water, the velocity potential of the two wave trains can be given as

\[ \phi_i = a_i \sigma_i k_i^{-1} e^{i k_i x} \sin \psi_i \]

where \( k_i = \left| k_i \right| \) is the wave number, \( \sigma_i \) the angular frequency, \( a_i \) the wave amplitude. The phase function, \( \psi_i \), is defined as \( \psi_i = k_i \cdot \overrightarrow{x} - \sigma_i t \) where \( t \) is time and \( \overrightarrow{x} \) is a vector representation of the two horizontal directions, \( x \) and \( y \). The origin of the coordinate system is located at the mean water surface with \( Z \), positive upward. "1 = 1" represents the first wave train and "2 = 2", the second wave train.

The horizontal velocity, \( u_i \), and acceleration, \( \ddot{u}_i \), of the wave trains can be computed as

\[ u_i = \nabla_H \phi_i = a_i \sigma_i e^{i k_i x} \cos \psi_i \]

and

\[ \ddot{u}_i = \frac{\partial u_i}{\partial t} = a_i \sigma_i^2 e^{i k_i x} \sin \psi_i \]

where \( \nabla_H \) is the spatial gradient in the horizontal direction.

Second-Order Waves

The second-order waves of the two deepwater wave trains are comprised of self-interaction and cross-interaction components. For a single irrotational wave in deep water, the velocity and acceleration of the self-interaction components vanish. Hence

\[ u_{20} = \nabla_H \phi_{20} = 0 \] (4)

and

\[ u_{12} = \nabla_H \phi_{12} = 0 \] (5)

and only the cross-interaction components remain. The velocity potential of the cross-interaction components, \( \phi_{12} \), is given as

\[ \phi_{12} = A e^{i \overrightarrow{k}_{12} x} \sin (\psi_1 - \psi_2) - B e^{i \overrightarrow{k}_{12} x} \sin (\psi_1 + \psi_2) \]

where

\[ A = \frac{2 a_1 a_2 \sigma_1 \sigma_2 (\sigma_1 - \sigma_2) \cos^2 \frac{1}{2} \theta}{\left| \sigma_1 - \sigma_2 \right|^2 + g \left| \overrightarrow{k}_1 - \overrightarrow{k}_2 \right|} \] (7)

\[ B = \frac{2 a_1 a_2 \sigma_2 \sigma_1 (\sigma_1 + \sigma_2) \sin^2 \frac{1}{2} \theta}{\left| \sigma_1 + \sigma_2 \right|^2 + g \left| \overrightarrow{k}_1 + \overrightarrow{k}_2 \right|} \] (8)

Here \( \theta \) is the angle between the two wave trains (see Figure 1) and \( g \) is the acceleration of gravity.
The horizontal velocity, \(u_{21}\), and acceleration, \(\ddot{u}_{22}\), are derived from Eq. (6) as

\[
\begin{align*}
u_{21} &= \left| k_1 - k_2 \right| A e^{-i k_2 z} \cos(\psi_1 - \psi_2) \\
&- \left| k_1^* + k_2^* \right| B e^{i k_2 z} \cos(\psi_1 + \psi_2)
\end{align*}
\]

and

\[
\begin{align*}\ddot{z}_{21} &= (\sigma_2 - \sigma_1) \left| k_1 - k_2 ight| A e^{-i k_2 z} \sin(\psi_1 - \psi_2) \\
&- (\sigma_1 + \sigma_2) \left| k_1^* + k_2^* \right| B e^{i k_2 z} \sin(\psi_1 + \psi_2)
\end{align*}
\]

Third-Order Waves

The third-order waves of the two deepwater wave trains generate the resonant interaction components which grow in time. The velocity potential, \(\phi_{21}\), of the resonant interaction components is

\[
\phi_{21} = -\frac{P}{2 (2\sigma_1 - \sigma_2)} \epsilon |k_1 - k_2|^2 \cos(2\psi_1 - \psi_2)
\]

where Longuet-Higgins defines \(P\) as

\[
P = (a_1 k_j^2 a_2 k_j) e^{-i k_j z} \int P(\epsilon)
\]

and

\[
F(\epsilon) = \left[ 1 + \frac{1}{2} \epsilon^2 + \frac{1}{2} \epsilon^3 \left( 1 - 4 \epsilon^2 \right) \right] \left( 1 + \frac{4 \epsilon}{\epsilon - (6 + \epsilon^2)^{1/2}} \right)
\]

\[
\epsilon = \frac{\sigma_2 - \sigma_1}{\sigma_1}
\]

\(P(\epsilon)\) is called the coupling factor. Eq. (12) is a compact form of Eq. (2.2) in Longuet-Higgins' paper.

Based on the work of Longuet-Higgins [5], the resonant interactions occur when the two deepwater wave trains give a locus of a "figure of eight" (see Figure 2). Positive values of \(\epsilon\) correspond to points on the right-hand loop, and negative values of \(\epsilon\) to points on the left-hand loop.
The dynamic response of a fixed offshore structure with and without nonlinear wave interaction effects (i.e., second-order cross interactions and third-order resonant interactions) will be derived as follows:

First-Order Motions

The water particle velocity, \( \mathbf{u} \), of the two first-order waves is \( \mathbf{u}_1 + \mathbf{u}_2 \), and the water particle acceleration, \( \mathbf{a} \), is \( \mathbf{a}_1 + \mathbf{a}_2 \). If the wave crest phase of the two wave trains is selected, there is only drag force (i.e., the first term on the right-hand side of Eq. (18)). For the mean water level \( \zeta = 0 \), the structural response due to the drag force is derived by inserting Eq. (18) into Eq. (17). The result is

\[
\begin{align*}
\dot{\mathbf{x}}_1 &= \frac{\rho}{2\pi} C_D D \left( \mathbf{a}_1 - a_y \mathbf{a}_y \right) \\
\mathbf{a} &= \frac{\rho}{2M} \mathbf{a}_1 + \frac{\rho}{2M} \mathbf{a}_2 + \frac{\pi D^2}{4} \left( a_1 a_y + a_y a_1 \right) \\
\end{align*}
\]  

(19)

First-Second-Order Motions

The water particle velocity, \( \mathbf{u} \), for the first-order and second-order waves is \( \mathbf{u}_1 + \mathbf{u}_2 \), and the water particle acceleration, \( \mathbf{a} \), is \( \mathbf{a}_1 + \mathbf{a}_2 \). By selecting the wave crest phase of the two deep water wave trains and the mean water level position \( \zeta = 0 \), the structural response due to drag force is

\[
\begin{align*}
\dot{\mathbf{x}}_2 &= \frac{\rho}{2M} C_D D \left[ a_1 a_y + a_y a_1 \right] \\
\mathbf{a} &= \frac{\rho}{2M} \mathbf{a}_1 + \frac{\rho}{2M} \mathbf{a}_2 + \frac{\pi D^2}{4} \left( a_1 a_y + a_y a_1 \right) \\
\end{align*}
\]  

(20)

First, Second, and Third-Order Motions

If the third-order waves are involved in the wave force, \( \mathbf{u} \) becomes \( \mathbf{u}_1 + \mathbf{u}_2 + \mathbf{u}_3 \), and the water particle acceleration, \( \mathbf{a} \), is \( \mathbf{a}_1 + \mathbf{a}_2 + \mathbf{a}_3 \). By selecting the wave crest phase of the two deep water wave trains, it is found that the drag force of the third-order waves does not exist, but the inertia force does. In other words, the wave force at the wave crest phase contains the drag force of the first-order and second-order waves, and the inertia force of the third-order waves. The structural response due to this wave force at mean water level position \( \zeta = 0 \) is

\[
\begin{align*}
\dot{\mathbf{x}}_3 &= \frac{\rho}{2M} C_D D \left[ a_1 a_y + a_y a_1 \right] \\
\mathbf{a} &= \frac{\rho}{2M} \mathbf{a}_1 + \frac{\rho}{2M} \mathbf{a}_2 + \frac{\pi D^2}{4} \left( a_1 a_y + a_y a_1 \right) \\
\end{align*}
\]  

(21)

Nonlinear Wave Interaction Effects

The comparisons of structural response with and without nonlinear wave interactions are based on two different categories: (1) at the same wave phase and (2) of the same type of wave force.

The first category consists of several comparisons. If the wave crest phase is selected, the ratio of the structural response with and without the second-order waves can be defined as \( R_{\mu} \). Dividing Eq. (21) by Eq. (19), \( R_{\mu} \) becomes

\[
R_{\mu} = \left( \frac{\left| \mathbf{k} - \mathbf{k}_2 \right| + \left| \mathbf{k} + \mathbf{k}_2 \right|}{\left| \mathbf{k}_1 \right| + \left| \mathbf{k}_2 \right|} \right)^2
\]  

(23)

By choosing the mean water level phase, the ratio of the structural response with and without the third-order waves is denoted as \( R_{\nu} \) and is derived using Eqs. (23) and (19)

\[
\begin{align*}
R_{\nu} &= \frac{1}{\left| \mathbf{k}_1 \right| + \left| \mathbf{k}_2 \right|} \left[ \left( \mathbf{k}_{\mu} \mathbf{a}_1 + \mathbf{a}_1 \mathbf{a}_2 \right) \mathbf{a} + \left( \mathbf{k}_{\nu} \mathbf{a}_1 + \mathbf{a}_1 \mathbf{a}_2 \right) \mathbf{a} \right] \\
&= \left( \frac{\left| \mathbf{k}_1 \right| + \left| \mathbf{k}_2 \right|}{\left| \mathbf{k}_1 \right| + \left| \mathbf{k}_2 \right|} \right)^2
\end{align*}
\]  

(24)

By choosing the mean water level phase, the ratio of the structural response with and without the third-order waves is denoted as \( R_{\delta} \). From Eqs. (20) and (22), it is

\[
R_{\delta} = \frac{1}{\left| \mathbf{k}_1 \right| + \left| \mathbf{k}_2 \right|} \left[ \left( \mathbf{k}_{\delta} \mathbf{a}_1 + \mathbf{a}_1 \mathbf{a}_2 \right) \mathbf{a} + \left( \mathbf{k}_{\nu} \mathbf{a}_1 + \mathbf{a}_1 \mathbf{a}_2 \right) \mathbf{a} \right] \\
= \left( \frac{\left| \mathbf{k}_1 \right| + \left| \mathbf{k}_2 \right|}{\left| \mathbf{k}_1 \right| + \left| \mathbf{k}_2 \right|} \right)^2
\]  

(25)
The ratio of the structural response with and without the third-order waves is \( R_{sm} \), which is derived from Eqs. (20) and (24) as

\[
R_{sm} = 1 + \left( \frac{1}{\sigma_1 \sigma_2^2 + \sigma_2 \sigma_1^2} \right) \left( \sigma_1 - \sigma_2 \right) \left[ \vec{k}_1 - \vec{k}_2 \right] [A]
\]

\[- (\sigma_1 + \sigma_2) [2\vec{k}_1 - \vec{k}_2] B + \left( \frac{C_D}{C_M} \right) \left( \frac{2}{\pi D} \right) \]

\[
\left[ \frac{P}{2(\sigma_1 - \sigma_2)^2} \right] \left[ 2\vec{k}_1 - \vec{k}_2 \right]^2
\]

\[
\left( \frac{t - \frac{2\lambda}{\omega_0}}{\frac{2\omega_0^2}{(l - 2\lambda)} \left( \vec{k}_1 \right)} \right) \right] \right) \right)
\]

(28)

Concerning the second category, if the drag force is focused, the ratio of the structural response with and without the second-order waves is represented by Eq. (25). The ratio of the structural response due to the third-order wave effects is denoted as \( Q_{sm} \), which can be derived using Eq. (19) and the first term on the right-hand side of Eq. (24)

\[
Q_{sm} = \left( \frac{P}{2(\sigma_1 - \sigma_2)^2} \right) \left( \frac{2\vec{k}_1 - \vec{k}_2}{\vec{k}_2} \right)^2
\]

\[
\left( \frac{t - \frac{2\lambda}{\omega_0}}{\frac{2\omega_0^2}{(l - 2\lambda)} \left( \vec{k}_1 \right)} \right) \right) \right)
\]

(29)

Similarly, the inertia force, Eq. (27) gives the ratio of the structural response due to the second-order wave effects. \( Q_1 \) will be defined as the ratio of the structural response due to third-order effects. By using the second term of the right-hand side of Eq. (23) and Eq. (20), \( Q_1 \) becomes

\[
Q_1 = \left( \frac{P}{2(\sigma_1 - \sigma_2)^2} \right) \left( \frac{2\vec{k}_1 - \vec{k}_2}{\vec{k}_2} \right)^2
\]

\[
\left( \frac{t - \frac{2\lambda}{\omega_0}}{\frac{2\omega_0^2}{(l - 2\lambda)} \left( \vec{k}_1 \right)} \right) \right) \right)
\]

(30)

where the "-" sign of the second term of Eq. (23) has been eliminated because only the magnitude is concerned.

4. RESULTS

To show the ratio of structural response with and without the presence of nonlinear wave interactions, some characteristic values of waves and structures have to be selected. The wave characteristic values chosen are \( \sigma_1 = 5 \text{ ft.}, \sigma_2 = 3 \text{ ft.}, \omega_0 = 0.628 \text{ rad/sec} \) (the dominant wave frequency in a typical hurricane), \( \sigma_2 = 0.344, \text{ and } \omega_0 = 1.3 \text{ rad/sec} \) (the natural frequency of the Cognac platform installed in 1025 ft. of water, see Sterling et al. [7]). The ratios, \( R_{sm} \) and \( R_{sm}' \), are found to be almost equal to unity. Hence, the second-order waves only produce a small modification to the structural motion. However, the structural motion due to the third-order effect is found to be significant.

Figure 3 shows the variation of \( R_{sm} \) with respect to the angle between the two wave trains, \( \theta \), for time scales "T" 50 and 100 times the wave period of the first wave train (i.e., \( T_1 \)). It can be seen that for \( \epsilon > 0 \) (i.e., \( \sigma_2 > \sigma_1 \)), the ratio of the structural response with and without the third-order waves at wave crest phase is smaller than for \( \epsilon > 0 \) (i.e., \( \sigma_2 > \sigma_1 \)). The magnitude of \( R_{sm} \) is less than unity. In other words, the structural response with the effect of the third-order waves at wave crest decreases with increasing the time scales. It is because the direction of the horizontal acceleration of the third-order waves is opposite to that of the horizontal velocities of the first and second-order waves.

A minimum value of \( R_{sm} \) occurs at \( \theta = \pi \) when \( \sigma_1 \) is smaller than \( \sigma_2 \). The minimum values are 0.93 for time scales 50 times the wave period of the first wave train and 0.85 for time scales 100 times the wave period.

The variation of \( R_{sm} \) versus the angle between the two wave trains, \( \theta \), is shown in Figure 4. By using the same time scales as \( R_{sm} \), the ratio, \( R_{sm}' \), of the structural response with and without the third-order waves at mean water level phase, for \( \epsilon < 0 \) is greater than that for \( \epsilon > 0 \). The structural response with the effect of the third-order waves at mean water level phase increases when the time scales increases. As \( \sigma_1 \) is smaller than \( \sigma_2 \), maximum values of \( R_{sm} \) occurring at \( \theta = 0 \), are 1.19 and 1.75 for the cases of \( T_1 = 50 \) and 100 wave period of the first wave train.

The ratio, \( Q_{sm} \), is shown plotted against \( \theta \) in Figure 5. It is greater when \( \epsilon < 0 \), and has a maximum near \( \theta = \pi \). As the time scales are 50 and 100 times the wave period of the first wave train, the maximum values of \( Q_{sm} \) are 0.052 and 0.21. \( Q_1 \) against \( \theta \) is shown in Figure 6. Like \( Q_{sm} \), it is greater over the range of \( \epsilon < 0 \) and the maximum occurs at \( \theta = 17^\circ \). The maximum values are 0.26 for \( T_1 = 50 \) and 0.53 for \( T_1 = 100 T_1 \).

The results show that the accretion of the structural response at mean water level phase is larger than the reduction of the structural response at wave crest phase. It is worthwhile to discuss whether the structural response due to the third-order waves at mean water level phase would be larger than that at wave crest phase. First, the results show that the structural response due to the drag force at mean water level (third-order waves) compared to that at wave crest phase (first-order waves), i.e., Figure 5, is much smaller than the structural response due to the inertia force at wave crest phase (third-order waves) compared to that at mean water level (first-order waves), i.e., Figure 6. Therefore, the real variation of the structural response at mean water level phase would not be as significant as the \( R_{sm} \) shown. Secondly, the magnitude of the structural response at wave crest phase is greater due to the additional submergence of structures at the wave crest phase (a contribution not included in the present results). Therefore, the structural response at wave crest phase is still larger than that at the mean water level phase.

5. CONCLUSIONS

The effect of the nonlinear wave interactions of deepwater waves play an important role on the dynamic response of fixed offshore structures. The results show that the second-order waves (cross interactions) only make a small modification to the structural response. The third-order waves (resonant interactions), however, produce a significant effect because they grow in time. The structural response due to the effect of the third-order waves at the wave crest phase increases with increasing time of resonant interactions. At mean water level phase, the structural response with the third-order wave effects increases as the time scale increases. For both wave phases, the variations of the structural response as \( \sigma_2 > \sigma_1 \) is larger than those as \( \sigma_2 < \sigma_1 \). Also, the variations vanish when the two wave trains are parallel or anti-parallel.

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