SAWTOOTH BIAS IN DIGITAL SPLIT BEAM TRACKERS

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ABSTRACT

Inaccurate dipole spacing modeling causes Sawtooth bias in digital split-beam trackers. This paper presents a unified treatment of the generalized dipole spacing model for a complex array and discusses the parametric dependence in detail. The study conducted revealed that in addition to conventional array/beamformer parameters, the dipole spacing depended on "quadrature" angular offset from boresight and signal-to-noise ratio. The former resulted from an effective phase shading, induced by discrete beamformer delays. The latter occurred with clipped beamformer and appeared as an amplitude shading induced by the hydrophone beam pattern. The magnitude of these effects varied with signal spectrum.

1.0 INTRODUCTION

Modern passive sonar systems employ digital preformed beams to achieve simultaneous detection coverage of multiple azimuthal and Depression/Elevation (D/E) angles. To accomplish fine tracking of bearing (and/or D/E), right and left (or up and down) half-beams are presented to the angle tracker. The angle tracker actually estimates the time-delay difference (τ) between equivalent phase centers of the half-beam apertures. The time-delay difference can be converted to bearing offset from the boresight angle of the tracking beam (θ = cr/D), if the half-beam acoustic phase center separation (D) (or dipole spacing) and the speed of sound (c) are known. Existing systems use a constant dipole spacing value for each beam. Errors in dipole spacing values produce a bias in the bearing estimate for a preformed beam system. A Split-Beam Tracker (SBT) operating from a preformed beam system will switch from one beam to another as a target traverses a bearing sector. The effect of the switch is called beam handover and any bearing discontinuities that result at the beam switching point are known as beam handover errors (or Sawtooth bias errors) (References 1, 2, 3). Beam handover error due to an off nominal dipole spacing value is given by

\[ \text{Beam handover error} = - (\Delta D / (D + \Delta D)) \Delta \theta \]  

where dipole spacing error, \( \Delta D = \) (nominal dipole) - (true dipole), and \( \Delta \theta \) is the beam spacing. Depend-

ing on the array in question, the percentage change of the ratio \( \Delta D / (D + \Delta D) \) may vary by as much as 10 to 20 percent of its nominal; thus producing a maximum beam handover error of 10 to 20 percent of the beam spacing. This is unacceptable for most high-performance angle trackers. Figure 1 illustrates a typical beam handover error characteristic from a hypothetical array.

This paper discusses the general dipole spacing behavior of a complex array and the sensitivities of various parametric dependencies. These dependencies include: aperture shape, quadrature angular offset from boresight, signal-to-noise ratio (SNR) and beam formation characteristics.

Section 2.0 provides fundamental discussions of the expansion, contraction and invariant properties of dipole spacing through a simple, three-element array. Section 3.0 analyzes the dipole spacing behavior of a complex spherical array with a linear beamformer, using the method of successive equivalent dipole reduction. The coupling between D/E and bearing dependencies is also discussed. Finally, Section 4.0 extends the results to include the effect of a clipped beamformer.

2.0 DIPOLE SPACING OF A LINEAR ARRAY WITH SYMMETRICAL APERTURE PERTURBATION

Figure 2 depicts a half-aperture of a non-linear array (or a linear array with element distortion) whose complex radiation pattern (for a linear beamformer) can be written as (Reference 4)

\[ f(\theta, k) = \sum_{i=1}^{N} a_i e^{ik r_i \cdot \hat{n}_s} \]  

where \( a_i \) = ith element shading, \( r_i \) = radial vector to the ith element, \( \hat{n}_s \) = target pointing vector, \( k \) = \( \omega / c \) wave number, \( \omega \) = target frequency, \( c \) = speed of sound in water, \( N \) = number of half-aperture hydrophone elements, and \( \theta \) = target bearing off a Maximum Response Axis (MRA).

Note that \( r_i \cdot \hat{n}_s = x_i \sin \theta + y_i \cos \theta \), and let \( \hat{r}_i = k r_i \cdot \hat{n}_s \). Equation (2) can be written in terms of the effective amplitude and phase as:

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\[ Ae^{i\phi} = \sum_{i=1}^{N} a_i e^{i\phi_i} \]

where

\[ \phi = \tan^{-1}\left( \sum_{i=1}^{N} a_i \sin \phi_i \right) \]

\[ \Delta = \tan^{-1}\left( Q/I \right) \]

\[ A = \sum_{i=1}^{N} a_i \cos(\phi_i - \phi) \]

and \( I, Q \) are the in-phase and quadrature phase components. Furthermore, defining \( \phi = k(x_c \sin \theta + y_c \cos \theta) \), then \((x_c, y_c)\) is the phase center location. Thus, one can represent any complex array by a one-element array located at \((x_c, y_c)\). We note that phase center location is non-stationary with respect to different target bearings.

By representing each half-aperture by a one-element array, the full aperture can be represented as a dipole. Therefore, from Figure 3, one can write:

\[ \Delta \phi = k \Delta \sin \theta \]

where \( \Delta \phi = \phi^R - \phi^L \) is the phase difference between the right and left half-beams. Furthermore, one can compute the dipole spacing about the beam bore-sight by:

\[ D = 1/k \left| \frac{\partial \phi^R}{\partial \theta} \right|_{\theta=0} - \frac{\partial \phi^L}{\partial \theta} \right|_{\theta=0} \]

where

\[ \frac{\partial \phi}{\partial \theta} \bigg|_{\theta=0} = \sum_{i=1}^{N} a_i \cos \phi_i(0) \]

\[ \Gamma_0 = \left\{ 1 + \frac{S_0(C_0)}{c_0} \right\} \left\{ 1 + \frac{S_1(C_1)}{c_1} \right\} \]

and

\[ s_0 = \sum_{i=1}^{N} a_i \sin \phi_i(0), \quad c_0 = \sum_{i=1}^{N} a_i \cos \phi_i(0), \]

\[ s_1 = \sum_{i=1}^{N} a_i \sin \phi_i(0) \cdot \phi_i(0) \frac{\partial}{\partial \theta} \]

\[ c_1 = \sum_{i=1}^{N} a_i \cos \phi_i(0) \frac{\partial \phi_i(0)}{\partial \theta} \]

Note that \( \Gamma_0 \approx 1 \) for \( \phi_i(0) \ll 1 \) radian.

For a line array with symmetrical element perturbations (transversal to the array) Equation (7) reduces to

\[ D = 2 \sum_{i=1}^{N} a_i \cos(ky_i) x_i \sum_{i=1}^{N} a_i \cos(ky_i) \cdot \Gamma_0 \]

Equation (10) indicates that element phase perturbation alter the effective element shadings in the calculation of dipole spacing. We shall designate it as "effective dipole shading" to signify its association with dipole spacing calculation. Note that Equation (10) reduces to the geometric centroid separation between two half-beams for the case of no element distortions.

A three-element array is selected to illustrate the dipole spacing variation resulting from element phase perturbation. A three-element array is chosen with shadings \((1, a, 1)\) and inter-element spacings, \(d=20\) feet. For a mid-element vertical position distortion \(y\), Equation (10) gives the exact expression for the dipole spacing about broadside as

\[ D = 2a \left( \frac{1 + a \cos(ky)}{1 + a^2 + 2a \cos(ky)} \right) \]

The following cases of interest exist:

Case 1 - Dipole Spacing Expansion. Choose any positive \(a<1\); the dipole spacing will increase for either a positive or negative mid-element distortion. Figure 4 shows the dramatic increase in dipole spacing due to mid-element distortion.

Case 2 - Dipole Spacing Contraction. Choose any \(a>1\). The dipole spacing will decrease for either a positive or negative mid-element distortion. Figure 5 shows the comparable results of prediction (using Equation 11) and the actual simulation.

Case 3 - Unchanged Dipole Spacing. Let \(a=1\). Dipole spacing is then insensitive to mid-element distortion. This is indicated by Equation (11) and can be proven exactly by computing the half-beam phase difference.

3.0 DIPOLE SPACING CALCULATION OF A COMPLEX SPHERICAL ARRAY

Our study of the three-element array indicated that dipole spacing varies as a function of aperture shape, element phase perturbations, and element shadings. As indicated by our study, these factors remain valid for a more general class of arrays, for example, a spherical array. Actually our findings are valid for any complex array with symmetrical left/right or up/down apertures. Assuming we are interested in the horizontal (bearing) dipole spacing, we can calculate the dipole spacing values by first locating the phase center pairs for each row and then calculating the equivalent array phase center from all the row phase center pairs. Using this method we can reduce any complex array into a dipole. The required procedures are straightfor-
ward but tedious (Reference 1), thus only the results are stated here. For any complex array with symmetrical aperture, a good approximation for the dipole spacing for each preformed beam is given by

\[ D(\phi) = 2 \sum_{r=1}^{N} A_r(0) \cos \phi \cos(\theta_r(0) x_c(r)) \]

where \( \phi = \text{Depression/Elevation angle}, A_r(0) = \text{effective amplitude for elements in the rth row}, \phi_r(0) = \text{effective element phase of the rth row}, x_c(r) = x\text{-coordinate location of the rth row phase center}, \theta_0 = \text{as given in Equation (9)}, \text{and } N = \text{number of row of elements}.\]

The effective phase error \( \phi_r(0) \) is given by

\[ \phi_r(0) = \frac{\pi}{2} + \sum_{m=1}^{N} \left[ \sin \phi_m - \sin \phi_n \right] \]

where \( \phi, \phi_0 \) are the D/Es of the target signal and beam direction, respectively. Thus, we see that any D/E angular offset from the beam direction will modify the dipole spacing (1) as a general cosine of D/E reduction factor and (2) as a dipole shading as discussed in Section 2.0. Furthermore, the wave number, \( k \), is related to the signal frequency. Therefore, the dipole spacing will be a function of signal frequency as well. The resulting dipole spacing variation is quite pronounced (as shown in Figure 7), and is representative of a hypothetical spherical array. Finally, we want to point out that for a rectangular aperture, no dipole shading effects exist. This can be shown easily by noting (in Equation (12)) that \( x_c(r) \) is identical for each row. Therefore, Equation (12) reduces to

\[ D(\phi) = w/2 \times \cos \phi \]

where \( w \) is the width of the aperture. Thus, from the point of reducing boresight angular offset dipole spacing variation, a rectangular aperture is preferable.

4.0 EXTENSION TO A CLIPPED BEAMFORMER

The dipole spacing of a general complex array with a clipped beamformer will be a function of the SNR if a non-uniform power distribution exists across the receiving array. The above phenomena is explained as follows:

1. At low SNR, the clipped beamformer generally behaves as a linear beamformer in power. Therefore, any power distribution across the receiving array normally will be passed through the clippers in unmodified fashion except for a common scale factor.

2. At higher SNR, the effect of the clipping tendency to equalize the output power distri-

3. Equalization of the non-uniform received power distribution dictates a new value of dipole spacing. Since the amount of equalization is dependent on SNR the dipole spacing is dependent on SNR.

From the above, it is obvious that all factors which would result in a non-uniform power distribution across the receiving array of a clipped beamformer system will result in a dipole spacing dependency on SNR. One potential source of a non-uniform power distribution across the receiving array is the directional hydrophone element.

Mathematical Model

A mathematical model for the dipole spacing dependency on SNR can be most readily derived assuming a target signal(s) consisting of a sinusoidal tone in the presence of zero mean Gaussian noise (N). The phase difference estimated by the tracker is modeled as the phase difference between the mean signal trajectories at the output of the two tracking half-beams. Therefore, the phase difference at the output of the tracker is given by the equation

\[ \Delta \Phi(r) = \Phi^R - \Phi^L = \tan^{-1} \left( \frac{\hat{I}_R - \hat{Q}_R}{\hat{I}_L + \hat{Q}_L} \right) \]

where \( \hat{I}_R, \hat{Q}_R \) are the in-phase and quadrature-phase components of the mean signal trajectory from the right half-beam, and \( \hat{I}_L, \hat{Q}_L \) are the in-phase and quadrature-phase components of the mean signal trajectory from the left half-beam.

The received signal model at the \( i \)th hydrophone is given by

\[ r_i = s_i + \eta_i = e_i \sqrt{2}\text{SNR} \cos(\omega t - \phi_i) + \eta_i \]

where SNR is the linear signal-to-noise ratio, \( e_i \) is the directional amplitude shading of the \( i \)th element, \( \eta_i \) is zero mean Gaussian random variable with unity variance, \( \phi_i \) is the \( i \)th element phase shift due to time delay error.

The expected output \( \mean{c_i} \) of the \( i \)th hydrophone and hard clipper defined by \( |c_i| = 1 \) if \( r_i \geq 0; |c_i| = 0 \) if \( r_i < 0 \) is given below

\[ \mean{c_i} = 1/2(1 + \text{erf}(s_i)) \]

where \( \text{erf} \) is the error function defined by

\[ \text{erf}(x) = \frac{2}{\sqrt{\pi}} \int_{0}^{x} e^{-t^2/2} dt \]

From the above, the expected output \( \mean{c_i} \) of the hard clipper output can be written as:

\[ \mean{c_i} = \sum_{i=1}^{N} \hat{c}_i - N/2 = 1/2 \sum_{i=1}^{N} \text{erf}(s_i) \]
where \( N \) is the number of hydrophones in the half-beam and thus the in-phase and quadrature-phase components are given by

\[
I = \frac{\pi}{T} \int_0^T B(t) \cos \omega t \, dt \quad (19)
\]

\[
Q = \frac{\pi}{T} \int_0^T B(t) \sin \omega t \, dt \quad (20)
\]

where \( T = 2\pi/\omega \) is the period of the input signal.

Equations (19) and (20) can be transformed into

\[
I = \sum_{i=1}^{N} Y_i \cos \phi_i \quad Q = \sum_{i=1}^{N} Y_i \sin \phi_i \quad (21)
\]

where \( Y_i \) is defined as

\[
Y_i = \frac{1}{2T} \int_0^T \text{erf}(e_i^{\sqrt{2SNR}} \cos \omega t) \cos \omega \, dt
\]

and it can be shown (Reference 3) that

\[
Y_i = \sqrt{\pi} e_i^{\sqrt{2SNR}} \sum_{n=0}^{\infty} \lambda_n (B_i)^n
\]

where \( B_i = e_i^{\sqrt{2SNR}} \) and \( \lambda_n \) can be generated by the recursion

\[
\lambda_n + 1 = \frac{(n + 1)(n + 2)}{(n + 1)(n + 2)} \lambda_n \quad \lambda_0 = 1
\]

Recall that for a linear beamformer with element shading \( a_i \), the \( I \) and \( Q \) components are given by

\[
I = \sum_{i=1}^{N} a_i \cos \phi_i \quad Q = \sum_{i=1}^{N} a_i \sin \phi_i \quad (22)
\]

Thus, Equation (21) of the clipped beamformer looks very much like Equation (22) with one major exception; namely, the clipped beamformer causes an amplitude shading which is a function of SNR. This, in essence, is the underlying mechanism which produces the dipole spacing SNR dependency. To further elaborate on this important result, the clipper-induced amplitude shading, \( Y_i \), is plotted in Figure (8) as a function of effective signal amplitude (\( \sqrt{2SNR} \)). The behavior of \( Y_i \), both at low SNR and high SNR, is of particular interest. At low SNR, \( Y_i \) varies linearly with amplitude; however, at high SNR it behaves essentially as a half-wave rectifier. From the foregoing discussions the following conclusions can be made regarding the SNR dipole spacing dependency for the clipped beamformer.

A clipper is a power sensitive device which, at low SNR, provides a linear power amplification. At higher SNR, it saturates at a constant level regardless of input power strength. Therefore, the resulting dipole spacing is subject to SNR variations if, and only if, there exists a nonuniform distribution of power across the array elements.

The expression for the half-beam phase difference is considerably simplified if one assumes a symmetrical array. Thus, Equation (15) can be simplified to

\[
\Delta \phi(T) = \phi^R - \phi^L = 2\tan^{-1}(Q/I) \quad (23)
\]

or

\[
\Delta \phi(T) = 2\tan^{-1}\left\{ \frac{\sum_{i=1}^{N} Y_i \sin \phi_i}{\sum_{i=1}^{N} Y_i \cos \phi_i} \right\} \quad (24)
\]

The equivalent dipole spacing is easily obtained from Equations (7) and (24). Namely,

\[
d = \frac{1}{\pi} \cdot \frac{3\Delta \phi}{\phi_0} \mid_{\phi_0 = 0} = \frac{\sum_{i=1}^{N} Y_i \cos \phi_i(0) x_i}{\sum_{i=1}^{N} Y_i \cos \phi_i(0)} \quad (25)
\]

where \( x_i \) is the \( x \)-component of the \( i \)th element location with respect to the array center and \( \phi_i(0) \) (as defined in Section 3.0) is the resulting phase error for an on-bearing MRA target.

5.0 SUMMARY AND CONCLUSIONS

The functional relationship between signal D/E arrival angle, the D/E MRA of the preformed beam selected by the tracker and dipole spacing has been closely examined. Dipole spacing is a critical parameter in the reconstruction of bearing (or D/E) from time delay measurements for a split-beam tracker that uses a preformed beam design. This study indicates that quadrature angular offset from the beam boresight could modify the dipole spacing values. In addition, for a clipped beamformer, dipole spacing has been shown to be a function of the target signal-to-noise ratio as well. Finally, sawtooth bias in digital split beam trackers may be corrected by providing on-line update of dipole spacing values (Reference 2).
REFERENCES

1. Ng, L. C. and R. A. LaTourette, AN/BQQ-5 Spherical Array Horizontal Dipole Spacing Variation Due to Deviation of Target Arrival Angle From the Vertical Maximum Response Axis (U), Naval Underwater Systems Center Technical Memorandum 791132, 1 October 1979. (CONFIDENTIAL)


Figure 6. Chosen Coordinate System for Spherical Array Dipole Spacing Calculation

Figure 7. Dipole Spacing Versus Off D/E MRA Target of a Spherical Array

Figure 8. Equivalent Gain Versus Normalized Signal Amplitude \( \text{AMP} \times \text{SIN} [\omega + \phi] + \text{N(t)} \text{ where } \text{VAR}[\text{N(t)}]=1 \)