ABSTRACT

Techniques for sampling and reconstruction of high frequency bandlimited (i.e. bandpass) waveforms are well known. One of the most common bandpass sampling techniques, introduced by Grace and Pitt [1] and known as quadrature sampling, involves sampling the signal and its quarter wavelength translation. Quadrature sampling allows for exact interpolation or reconstruction of the bandpass waveform as long as the sampling rate is a submultiple of the center frequency of the signal. In many cases of practical interest, such precise sampling may not be possible. In this case, Persons [2] addressed the problem of determining the ratio of the power in the error bands to the power in the signal band when quadrature sampling is employed using an arbitrary sampling frequency. The problem addressed here is a generalization of Persons' formula to include the effects of sampling timing errors when applying the Grace-Pitt interpolation formula. The reconstruction error is plotted as a function of sampling timing errors.

INTRODUCTION

Consider a bandpass waveform $f(t)$ of carrier $f_0$ and bandwidth $W$. Recall, quadrature sampling (Figure 1) requires samples of both the bandpass waveform $f(t)$ and its quarter-wavelength translation $f(t-1/(4f_0))$ for exact reconstruction. The Grace-Pitt interpolation formula [1] is:

$$g(t) = \sum_{n=-\infty}^{*} f(n/f_g)s(t-n/f_g) + \sum_{n=-\infty}^{*} f(n/f_g+1/xf_0)s(t-n/f_g-1/xf_0)$$ (1)

where $x = 4$, allows an exact reconstruction of high frequency bandlimited waveforms with a minimum average sampling rate of $W$ samples/s (or $W/2$ samples/s/channel) whenever $2f_0 = kW$ ($k$ = integer). This was verified by Persons when he developed a quadrature sampling error formula [2].

For the general case, where $2f_0$ does not equal $kW$, the interpolation formula given by Eq (1) with $x = 4$ will not exactly yield $f(t)$. Persons showed a way to measure the amount of reconstruction error when using this equation for the general case. More specifically, he determined $Q$, the ratio of the power in the error band to the power in the signal band.

By following the same approach taken by Persons, it is possible to more fully develop the quadrature sampling error formula into a generalized formula in which the effects on $Q$ due to sampling timing errors can be investigated. We can show that timing is extremely critical when the sampling rate is equal to $W$ samples/s (average).

ANALYSIS

Beginning with the general interpolation formula of Eq (1), recall that the objective is to determine the power in the error relative to the power in $f(t)$.

We can express Eq (1) as

$$g(t) = \{f(t)\} \cdot \delta(t-n/f_g) * s(t)$$

$$+ \{f(t)\} \cdot \delta(t-n/f_g-1/xf_0) * s(t)$$ (2)

where $*$ denotes convolution.

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Taking the Fourier transform of Eq (2) yields

\[
G(f) = f_s S(f) \delta(f - n f_s) + \sum_{n=-\infty}^{\infty} \delta(f - n f_s) \exp(-j 2 \pi n f_s / x f_0) \quad (3)
\]

\[
G(f) = 2 f_s S(f) \{F(f) + (1/2) \sum_{n \neq 0} F(f - n f_s) [1 + \exp(-j 2 \pi n f_s / x f_0)] \}
\]

Let \(2 f_s S(f) = 1\) in the region where \(F(f)\) is nonzero and zero elsewhere,

\[
2 f_s S(f) = \text{rect}(f - f_0) / W + \text{rect}(f + f_0) / W \quad (5)
\]

This is equivalent to applying an ideal antialiasing filter to the bandpass signal. Solving for \(S(f)\)

\[
S(f) = (1/2 f_s) \{\text{rect}(f - f_0) / W + \text{rect}(f + f_0) / W\} \quad (6)
\]

The inverse Fourier transform of \(S(f)\) is

\[
s(t) = (W / f_s) \text{sinc}(W t) \cos(2 n f_0 t) \quad (7)
\]

Substituting Eq (6) into Eq (4) yields

\[
G(f) = F(f) + (1/2) \{\text{rect}(f - f_0) / W + \text{rect}(f + f_0) / W\} \sum_{n \neq 0} F(n - n f_s) / 1
\]

\[
+ \exp(-j 2 \pi n f_s / x f_0) \}\}
\]

Notice that \(G(f)\) is made up of \(F(f)\), the term we want, plus some extra terms. It is the relative power in these extra terms that will be determined. Because of the rect functions (antialiasing filter), only the terms giving bands in the vicinity of \(f_0\) will interfere.

Let \(2 f_0 / f_s = z + \tau = K\), where \(z\) is integer. Also, assume that \(f_s > W\), which is an obvious need or else the repeated spectral bands would overlap. Note that for \(2 f_0 / f_s = z\) integer, and \(x = 4\), there is only one band for \(n = z\) which overlaps \(F(f)\), but the amplitude term \(1 + \exp(-j \pi) = 0\).

In general, there will be two bands overlapping \(F(f)\), corresponding to the terms \(n = z, z + 1\). Define \(Q\) as the ratio of power in the extra bands to the power in \(F(f)\), and consider only positive frequencies. Also, define

\[
a_z = (1/2) [1 + \exp(-j \pi f_s z / x f_0)] \quad (9)
\]

\[
a_{z+1} = (1/2) [1 + \exp(-j \pi f_s (z + 1) / x f_0)] \quad (10)
\]

\[
B_z = \text{bandwidth of \(z\)th term} \quad (11)
\]

\[
B_{z+1} = \text{bandwidth of \((z+1)\)st term} \quad (12)
\]

Since we have applied the antialiasing filter, the numbers \(B_z\) and \(B_{z+1}\) are determined by the overlap with \(\text{rect}(f - f_0) / W\). Let the power in \(F(f) = V\); then from Eqs (8-12)

\[
Q = a_z [B_z / V] + a_{z+1} [B_{z+1} / V] \quad (13)
\]

where \(B_z\) and \(B_{z+1}\) must be greater than zero, else they are taken as zero.

Now, from Eq (9)

\[
|a_z|^2 = (1/4) [1 + \exp(-j \pi f_s z / x f_0)]^2 \quad (14)
\]

but \(\tau = 2 f_0 / f_s\) and \(z = [K]\), where \([\ ]\) is the greatest integer function.

\[
|a_z|^2 = (1/4) [1 + \exp(-j \pi f_s (z + 1) / x f_0)]^2 \quad (15)
\]

\[
|a_{z+1}|^2 = \cos^2(2 \pi \{z + 1 / x K\}) \quad (16)
\]

Likewise,

\[
|a_{z+1}|^2 = (1/4) [1 + \exp(-j \pi f_s (z + 1) / x K)]^2 \quad (17)
\]

\[
|a_{z+1}|^2 = \cos^2(2 \pi \{z + 1 / K\} / x K) \quad (18)
\]

The center of the \(z\)th band is at

\[
f_z = (z f_s - f_0) = \{K f_s - f_0\} \quad (19)
\]

and the center of the \((z+1)\)st band is at

\[
f_{z+1} = (z f_s - f_0) = \{(z+1) f_s - f_0\} \quad (20)
\]
Now determine $B_z/W$.

$$B_z/W = (f_z + W - f_o)/W$$  (21)

$$B_z/W = f_z/W + 1 - f_o/W$$  (22)

$$B_z/W = f_z/W + 1/2 - (f_o/W - 1/2)$$  (23)

Substituting Eq (19) into Eq (23)

$$B_z/W = f_z/W + 1 - (f_o/W - 1/2)$$  (24)

Since $f_o = f_sK/2$ ; $K = 2f_o/f_s$

$$B_z/W = 1 - (f_s(K-K))/W$$  (25)

Likewise,

$$B(z+1)/W = 1 - (f_s(1-K+K))/W$$  (26)

Substituting into Eq (13) and using the definition $K = z + x = (K) + x$ and $L = f_s/W$

$$Q(K,L) = \cos^2(2\pi(K/L))((1-Lr))$$

$$+ \cos^2(2\pi((K+1)/L))((1-L(1-r)))$$  (27)

where {{1}} indicates the quantity is to be taken as zero if it is negative. Eq (27) holds for $L \leq 2f_o/W + 1$; for larger $L$, the error is zero.

This is the desired result. Note that substituting $x = 4$ into Eq (27) will yield the quadrature sampling error formula developed by Persons.

Given $f_o$ and $W$, $Q$ can be plotted as a function of $K$ for any $L$. As mentioned, Eq (27) provides us with the flexibility to investigate the effects of sampling timing error. If we use Persons' example we can establish it as a baseline (he assumed no timing error). In other words, we'll assume that we've sampled $f(t)$ and $f(t-1/4f_o)$ with no timing error. This is the case where $x = 4$. Letting $L = f_s/W = 1$ we have

$$Q(K,L+1) = \cos^2((z/2K))((1-r))$$

$$+ \cos^2((z+1)/2K))((1-r))$$  (28)

which is plotted in Figure 2. Figure 2a shows the full scale of $Q$ for $0 \leq Q \leq 1.0$ where $1.0 = 100\%$ error. In part b the $Q$ scale was expanded to emphasize the shape of the error plot. Notice that $Q$ is zero for $K = integer$ and approaches zero for narrowband systems (large values of $K$).

The interesting question is, what happens to $Q$ if we have a timing error? In other words, we think we're sampling $f(t)$ and $f(t-1/4f_o)$, but we're actually sampling $f(t)$ and $f(t-1/4f_o)$ where $x$ does not equal 4. Since quadrature sampling requires sampling the bandpass signal and its quarter-wavelength translation, we expect some reconstruction error. The next few figures will show that there is indeed a reconstruction error and this error can be quite significant. For instance, if $x = 2$ then the bandpass signal and its half-wavelength translation are being sampled and the reconstruction error approaches $100\%$ (the signal cannot be reconstructed).

$Q$ is plotted in Figure 3a for $x = 4.0, 3.9, 3.8, 3.7, 3.6, 3.5$ and $0 \leq Q \leq 1.0$ and once again in Figure 3b with $0 \leq Q \leq 0.16$. The individual plots for the various values of $x$ are more easily distinguishable in part b. Notice as the timing error increases, or $|x-4.0|$ grows larger, that $Q$ increases. Figure 4 shows the case where the timing error has increased to the point where we are sampling the bandpass waveform and its half-wavelength ($x = 2.0$). In this figure, $Q$ is plotted with $x = 4.0, 3.8, 3.5, 3.0, 2.5, 2.0$. As before, $Q$ approaches $100\%$ as $x$ approaches 2.0. Figure 4b is a three-dimensional surface plot of part a.

To summarize, consider an example with three high frequency bandpass signals with center frequencies of $f_1 = 40$ MHz, $f_2 = 500$ MHz, and $f_3 = 1$ GHz. Once again let $L = L = f_s/W$. The timing errors, $T_E$, is a function of $x$:

$$T_E = T(1/4 - 1/x)$$  (29)

where $T$ is the period of the corresponding bandpass signal.

In Figure 4, a value of $x = 2.0$ results in a $Q$ of almost $1.0$ (or $100\%$). Referring to Figure 5, and putting this in terms of a timing error, we see that for $f_1$, a $T_E$ of only $6.25nS$ yields a $Q$ of $100\%$. Similarly, $Q = 100\%$ when $T_E = 500$ pS and $250$ pS for $f_2$ and $f_3$, respectively. The higher the center frequency, the more precise the timing must be to keep $Q$ small.
Typical communication signals will be sampled at the IF stages of the receiver, so $T_E$ for $f_1$ is representative of the type of timing precision needed for quadrature sampling when $f_S = W$ samples/s. Referring again to Figure 4 and looking at the column for $f_1$ in Figure 5, where $x = 3.8$, $T_E = 329 \text{ pS}$ and $Q$ is relatively small for large values of $K$. A value of $x = 3.0$ represents a 2 nanosecond timing error and a $Q$ that levels off at 0.3 (30%). Obviously, timing must be kept to nanosecond precision or it won't be possible to reconstruct the signal from the samples.

The next question is, what can be done to get around this timing problem? The answer is to sample at a higher rate. Figures 6-7 are plots of $Q$ for $f_S = 2.0W$ and $f_S = 5.0W$ respectively. As the sampling rate grows large compared to the bandwidth the error terms approach zero. In Figure 6, $Q$ is negligible for $x = 4.0$ (no timing error), 3.8 and 3.5. The only significant $Q$ terms are for $x = 3.0$, 2.5 and 2.0. Increasing sampling rates will reduce the $Q$ terms even further.

**SUMMARY**

To summarize, quadrature sampling requires very precise timing. Even with center frequencies less than 100 MHz, nanosecond precision is required. If such precision is not possible, higher sampling rates will be needed to overcome the reconstruction error.

**REFERENCES**


Figure 3. $Q$ vs $K$ with $L = 1.0$ - Small Timing Errors

Figure 4. $Q$ vs $K$ with $L = 1.0$ - Large Timing Errors