ROBUST CONTROL THEORY OF LINEAR TIME-VARYING
MULTIDIMENSIONAL DYNAMICAL SYSTEMS

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Abstract

The main purpose of this work is to treat aspects of the problem of robust control theory and mathematical modelling and simulation of large time-varying multidimensional dynamical systems. Such systems may also include certain nonlinear control theory problems with respect to construction and synthesis of feedback control laws. Pipeline processing may be used in considering robust feedback control laws where parameters are introduced into the systems, which may also include singular systems. In order to accomplish a large scale computer simulation of such dynamical systems one needs to treat both discretized and continuous versions of the same system.

Introduction

Robust control theory is concerned with the problem of analyzing and synthesizing control systems that provide an acceptable level of performance where model uncertainties exist, since mathematical models of physical systems are usually never exact due to the presence of multiparameters. The need to be able to design such feedback controls is very important in physical systems. Usually a model will have significant structural information about the interconnection of components and subsystems but less information about their integrated system performance.

A treatment of linear time varying dynamical systems of the form

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]

(*)

is given for the time-varying systems of \( r \) first order coupled linear differential equations above where \( x(t) \) is an \( r \)-vector, a state vector, \( u(t) \) is an \( n \)-vector of control inputs or forcing functions and \( y(t) \) is a \( p \)-vector of outputs. The main purpose of this work is to establish sufficient conditions for the existence of a feedback control law for given matrices \( A(t), B(t), C(t), D(t) \).

Two cases of (*) above are of interest. Case (i) where all matrices have elements which are continuous functions of \( t \) for \( t \geq 0 \) and case (ii) where all matrices have elements which are holomorphic functions of a single complex variable \( z \) for \( z \) belonging to bounded domains in the complex \( z \)-plane. Also an algorithm is given to obtain such feedback control laws for such cases. Results obtained generalize the classical problem where the matrices in (*) are constant matrices.

Singular systems of the form

\[
\begin{align*}
E(\theta)x(t,\theta) &= A(\theta)x(t,\theta) + B(\theta)u(t,\theta) \\
y(t,\theta) &= C(\theta)x(t,\theta) + D(\theta)u(t,\theta)
\end{align*}
\]

(**)

are also considered. The elements of the matrices \( E(\theta), A(\theta), B(\theta), C(\theta), D(\theta) \) belong to the ring of polynomials \( f(e) \), where \( \theta = (\theta_1, \theta_2, \ldots, \theta_n) \) is a multiparameter, with coefficients belonging to the field of complex numbers. \( f(e) \) may also be of the form \( f(e) = a(e)/b(e) \) for \( b(e) \neq 0 \), where \( a(\theta), b(\theta) \) are such polynomials. \( E(\theta) \) may be a singular matrix. The impulsive modes that appear in the free-response of such systems when arbitrary initial conditions are permitted will also be considered.

TIME VARYING DYNAMICAL SYSTEMS

In this section a treatment of the time-varying dynamical system of the form

\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t)
\end{align*}
\]

will be given. The dynamical systems will consist of \( r \) first order coupled linear differential equations above where \( x(t) \) is an \( r \)-vector, a state vector, \( u(t) \) is an \( n \)-vector of control inputs or forcing functions and \( y(t) \) is a \( p \)-vector of outputs. The main purpose of this section is to establish sufficient conditions for the existence of a feedback control law for given matrices \( A(t), B(t), C(t), D(t) \).

Two cases of the above system will be treated. Case (i) where all matrices have elements which are continuous functions of \( t \) for \( t \geq 0 \) and case (ii) where all matrices have elements which are holomorphic functions of a single complex variable \( z \), for \( z \) belonging to bounded domains in the complex \( z \)-plane. Also an algorithm is given to obtain such feedback control laws. Results obtained generalize the
classical problem where the matrices are constant matrices.

Example. Singular Systems.
Consider the time-invariant system of first order coupled linear differential equations
\[
\begin{align*}
E \dot{x}(t) &= Ax(t) + Bu(t) \\
y(t) &= Cx(t) + Du(t) + Fz(t), 
\end{align*}
\]
where \(x(t)\) is an \(r\)-vector of internal variables, \(u(t)\) is an \(m\)-vector of control inputs or forcing functions and \(y(t)\) is a \(p\)-vector of outputs.
Let \(E\), \(A\), \(B\), \(C\), \(D\), \(F\) denote the vector space of \(m\times r\), \(r\times m\), \(m\times p\), \(r\times p\), \(p\times r\), \(p\times p\) matrices having real or complex elements. \(E\) may be a singular matrix. Let \(E,A \in \mathbb{R}^{m \times r}\), \(B,C \in \mathbb{R}^{r \times m}\), and \(D,F \in \mathbb{R}^{r \times p}\), where \(m\) and \(p\) may be singular matrices. Let \(E,A \in \mathbb{R}^{m \times r}\), \(B,C \in \mathbb{R}^{r \times m}\), and \(D,F \in \mathbb{R}^{r \times p}\), where \(m\) and \(p\)

The above system may be written as follows:
\[
\begin{bmatrix} E & -B \\ F & D \end{bmatrix} \begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} Ax(t) \\ y - Cx(t) \end{bmatrix}
\]

This equation has a solution \(u(t)\) if and only if
\[
\begin{bmatrix} E & -B \\ F & D \end{bmatrix} \begin{bmatrix} A \\ y - C \end{bmatrix} = 0
\]
holds where the symbol \(-\) denotes a \((1-2)\) generalized inverse of a matrix where
\[
\begin{bmatrix} I_n & -B \\ 0 & I_p \end{bmatrix} - \begin{bmatrix} E & -B \\ F & D \end{bmatrix} \begin{bmatrix} A \\ y - C \end{bmatrix} = 0
\]

Then in this case a general solution of \(u(t)\) is given as follows:
\[
\begin{bmatrix} \dot{x}(t) \\ u(t) \end{bmatrix} = \begin{bmatrix} E & -B \\ F & D \end{bmatrix} \begin{bmatrix} A \\ y - C \end{bmatrix} + \begin{bmatrix} I_n & -B \\ 0 & I_p \end{bmatrix} z(t)
\]

for arbitrary \(z(t)\).
Consider the specific equations:
\[
\begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix} = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix} \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + \begin{bmatrix} 0 \\ 1 \end{bmatrix} u(t)
\]

\[
y(t) = (1 \ 0) \begin{bmatrix} x_1(t) \\ x_2(t) \end{bmatrix} + Du(t) + (f_{11} \ f_{12}) \begin{bmatrix} \dot{x}_1(t) \\ \dot{x}_2(t) \end{bmatrix}
\]

and \(F = (f_{11} \ f_{12})\). Consider the matrix \(S\) associated with the above system of equations:
\[
S = \begin{bmatrix} E & -B \\ F & D \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & -1 \\ f_{11} & f_{12} & 0 \end{bmatrix}
\]

In order to determine a \((1-2)\) generalized inverse of \(S\) above making use of a method of John Jones, Jr.\[3\] as follows:
\[
\begin{bmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \end{bmatrix} f_{12} D f_{11} = 0
\]

and \(F = (f_{11} \ f_{12})\). Consider the matrix \(S\) associated with the above system of equations:
\[
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} f_{12} D f_{11} = 0
\]

and \(F = (f_{11} \ f_{12})\). Consider the matrix \(S\) associated with the above system of equations:
\[
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} f_{12} D f_{11} = 0
\]

and \(F = (f_{11} \ f_{12})\). Consider the matrix \(S\) associated with the above system of equations:
\[
\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & -1 & 0 \end{bmatrix} f_{12} D f_{11} = 0
\]
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
0 & -1 & 0 \\
0 & 1 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
-f_{12} & D & 1 \\
-f_{11} & f_{11} & f_{11} \\
-f_{11} & f_{11} & f_{11}
\end{bmatrix}
\begin{bmatrix}
0 & 0 & 1 \\
1 & 0 & 0 \\
0 & 1 & 0
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
f_{11} & f_{12} & f_{13} \\
f_{21} & f_{22} & f_{23} \\
f_{31} & f_{32} & f_{33}
\end{bmatrix}
\begin{bmatrix}
-f_{11} & f_{12} & 1 \\
-f_{11} & f_{12} & 1 \\
-f_{11} & f_{12} & 1
\end{bmatrix}
\begin{bmatrix}
x_1(t) \\
x_2(t) \\
x_3(t)
\end{bmatrix}
\begin{bmatrix}
-x_2(t) \\
-x_3(t) \\
-x_3(t)
\end{bmatrix}
\]

Finally, an inverse system is given below:
\[
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
0 & 1 & 0 \\
0 & 0 & 1
\end{bmatrix}
\begin{bmatrix}
1 & 0 & 0 \\
1 & 0 & 0 \\
1 & 0 & 0
\end{bmatrix}
\begin{bmatrix}
f_{11} & f_{12} & f_{13} \\
f_{21} & f_{22} & f_{23} \\
f_{31} & f_{32} & f_{33}
\end{bmatrix}
\begin{bmatrix}
-x_2(t) \\
-x_3(t) \\
-x_3(t)
\end{bmatrix}
\]

A linear dynamical system can be represented by the equations:
\[
\begin{align*}
\dot{x}(t) &= A(t)x(t) + B(t)u(t) \\
y(t) &= C(t)x(t) + D(t)u(t), \quad t \geq 0
\end{align*}
\]
where \(A, B, C, D\) are matrices of continuous functions of \(t\) for \(t \geq 0\). Assume that \(D(t)\) has constant rank \(m\) for \(t \geq 0\) and that \(D(t)D^\top(t) = I_m\) for any \(1\)-generalized inverse of \(D(t)\). In order to determine a control law we first obtain a \(1\)-generalized inverse system associated with the above system.

The above system can be written in the following form:
\[
\begin{bmatrix}
I_n & -B(t) & [A(t)x(t)]_t \\
0 & D(t) & [y(t) - C(t)x(t)]_t
\end{bmatrix}, \quad t \geq 0
\]
and we need to solve the above system for the vector:
\[
\begin{bmatrix}
\dot{x}(t) \\
u(t)
\end{bmatrix}
\]
If \(D^\top(t)\) is any \(1\)-generalized inverse of \(D(t)\) then
\[
\begin{bmatrix}
I_n & -B(t) & I_n \\
0 & D(t) & 0
\end{bmatrix}
\begin{bmatrix}
I_n & B(t)D^\top(t) & I_n \\
0 & D(t) & 0
\end{bmatrix}
\begin{bmatrix}
-x_2(t) \\
x_2(t) \\
-x_2(t)
\end{bmatrix}
\]
is a general inverse of associated with the dynamical system (*)

Now (**) has a general solution of the form

\[ P(t) = (A(t)B(t) + D(t))X(t) + B(t)D(t)h(t) \]

where \( h(t) \) is an arbitrary \( p \times m \) matrix. The consistency condition is satisfied by any choice of the \((p-m)\) arbitrary elements of the \( p \times m \) matrix \( [I_p - D(t)D(t)]h(t) \).

If \( D(t) \) is any \( 1 \)-generalized inverse of \( D(t) \) the general form for all \( 1 \)-generalized inverses of \( D(t) \) is as follows:

\[ D^-(t) = D^-t(t) + K(t) - D(t)D(t)K(t) \]

where \( K(t) \) is an arbitrary \( p \times m \) matrix. Any choice of \( K(t) \) will give an \( 1 \)-generalized inverse of (**), however important properties of the inverse system will depend upon such a \( K(t) \) which is chosen, such as controllability, stability, and observability of the inverse system (***)

The consistency condition

\[
\begin{bmatrix}
I_n & B(t)D^-(t) - B(t)
\end{bmatrix}
\begin{bmatrix}
I_n & -B(t)
\end{bmatrix}
\begin{bmatrix}
I_n & B(t)D^-(t)
\end{bmatrix}
\begin{bmatrix}
0 & D(t)
\end{bmatrix}
\]

and so

\[ s(t) = \begin{bmatrix}
I_n & B(t)D^-(t)
\end{bmatrix}
\begin{bmatrix}
0 & D(t)
\end{bmatrix}
\]

is a \( 1 \)-generalized inverse of

\[ S = \begin{bmatrix}
I_n & -B(t)
\end{bmatrix}
\begin{bmatrix}
0 & D(t)
\end{bmatrix}
\]

associated with the dynamical system (*) above.

Now (***) has a general solution of the form

\[ \dot{x}(t) = \begin{bmatrix} A(t) - B(t)D(t)C(t) \end{bmatrix}x(t) + B(t)D(t)h(t) \]

\[ u(t) = \begin{bmatrix} -D(t)C(t) \end{bmatrix}x(t) + D(t)y(t) + \begin{bmatrix} [I_p - D(t)D(t)]h(t) \end{bmatrix} \]

for \( t \geq 0 \).

Consider the following problem:

\[ \begin{bmatrix}
\begin{bmatrix} (t^3+t^2-2t-3) & 4(t+1)^2 \\
(t+1)^2 & (t+1)^2 \\
\end{bmatrix} \\
\begin{bmatrix} (t^3+3t^2+6t+1) & (t+1)^2 \\
(t+1)^2 & (t+1)^2 \\
\end{bmatrix}
\end{bmatrix} \begin{bmatrix}
(t^2+t-1) \\
(t+1)
\end{bmatrix} + \begin{bmatrix} (t^2+t-1) \\
(t+1)
\end{bmatrix} x(t) + \begin{bmatrix} -t(t+1) \\
(t+1)
\end{bmatrix} y(t) = C(t) + D(t)u(t) \]

for \( t \geq 0 \).

Here:

\[ C_1 A(t)C_1 = \begin{bmatrix}
1 & 0 \\
(t+1) & (t+1)
\end{bmatrix} \begin{bmatrix}
1 & 0 \\
0 & 1
\end{bmatrix} \]

and

\[ C_2 A(t)C_2 = \begin{bmatrix}
(t^3+t^2-2t-3) & 4(t+1)^2 \\
(t+1)^2 & (t+1)^2 \\
\end{bmatrix} \begin{bmatrix}
(t^2+t-1) \\
(t+1)
\end{bmatrix} + \begin{bmatrix} (t^2+t-1) \\
(t+1)
\end{bmatrix} x(t) + \begin{bmatrix} -t(t+1) \\
(t+1)
\end{bmatrix} y(t) = C(t) + D(t)u(t) \]

for \( t \geq 0 \).
where
\[ B(t) \]
is chosen such that
\[ B(t) = Q^{-1}(t)A(t) - Q^{-1}(t)Q(t) \]
or
\[ A(t) = P^{-1}(t)B(t)P(t) - P^{-1}(t)P(t) \]
holds where the difference of the diagonals
\[ a_i(t) \text{ of } (C_1^{-1}A(t)C_1) \] and \[ b_i(t) \text{ of } (C_2^{-1}B(t)C_2) \]
satisfy the following:
\[ a_i(t) - b_i(t) = \frac{1}{(t+1)^2} \int_0^t [a_i(t) - b_i(t)] dt = 1 \]

Thus \( B(t) \) is chosen as Lyapunov equivalent to \( A(t) \). Such choice of \( B(t) \) is convenient for unbounded systems \( \dot{x}(t) = A(t)x(t) \) to track \( \dot{y}(t) = B(t)y(t) \) or for the two systems to be kinematically similar to each other, i.e.

\[ P(t) = Q^{-1}(t), \text{ or } A(t) \text{ and } B(t) \text{ are said to be L-equivalent to each other if both are defined for } t > 0 \text{ and if there exists a Lyapunov transformation } Q = Q(t) \text{ such that } B(t) = Q^{-1}(t)A(t)Q(t) - Q^{-1}(t)Q(t) \text{ and } A(t) = P^{-1}(t)B(t) \]

\[ P(t) - P^{-1}(t)P(t), \text{ where } P(t) = Q^{-1}(t)\text{ holds.} \]

If \( \dot{x}(t) = A(t)x(t) \) and \( \dot{y}(t) = B(t)y(t) \) and \( x(t) = Q(t)y(t) \) for some \( Q(t) \) in \( L(0,\infty) \) then the zero solution for \( y(t) \) is asymptotically stable, stable, or unstable whenever the zero solution \( x(t) \) is as such. This may be useful in choosing a \( B(t) \) matrix for a given \( A(t) \) matrix in (*)

In order to find an inverse system of the above example consider the following:
In this section singular multiparameter dynamical systems of the form
\[ E(\theta) \dot{x}(t, \theta) = A(\theta)x(t, \theta) + B(\theta)u(t, \theta) \]
\[ y(t, \theta) = C(\theta)x(t, \theta) + D(\theta)u(t, \theta) \]
will be considered, where the elements of the matrices \( A(\theta), B(\theta), C(\theta), D(\theta), F(\theta) \) belong to the ring of polynomials \( f(\theta) \) where \( \theta = (\theta_1, \theta_2, \ldots, \theta_m) \) is a multiparameter, having coefficients belonging to the field of complex numbers. \( f(\theta) \) may also be of the form \( f(\theta) = a(\theta)/b(\theta) \) for \( b(\theta) \neq 0 \), where \( a(\theta), b(\theta) \) are such polynomials, \( f(\theta) \) may also belong to the ring of holomorphic functions of a single complex variable \( z \) for \( z \) belonging to bounded regions in the \( z \)-plane. \( E(\theta), A(\theta), B(\theta), C(\theta), D(\theta) \) may also be singular matrices. The impulsive modes that appear in the free-response of such systems when arbitrary initial conditions are permitted will be considered.

Let \( \mathcal{M}(\theta) \) denote the vector space of multiparameter matrices as described above.

Let \( E(\theta), A(\theta) \in \mathcal{M}(\theta), B(\theta) \in \mathcal{M}(\theta), C(\theta) \in \mathcal{M}(\theta), D(\theta) \in \mathcal{M}(\theta) \), where \( \mathcal{M}(\theta) \).

Also suppose that \( D(\theta)D^{-1}(\theta) = I_m \) for any 1-generalized inverse of \( D(\theta) \) such that \( D(\theta)D^{-1}(\theta)D(\theta) = D(\theta) \) holds.

The main purpose of this section is to obtain a 1-generalized inverse system of (*) above. Such a system is of the form:
\[ \hat{x}(t, \theta) = A_1(\theta)x(t, \theta) + B_1(\theta)u(t, \theta) \]
\[ u(t, \theta) = C_1(\theta)x(t, \theta) + D_1(\theta)\hat{y}(t, \theta) \]
Equation (**) may be written in the following form:
\[ [E(\theta) - D(\theta)] \begin{pmatrix} \hat{x}(t, \theta) \\ u(t, \theta) \end{pmatrix} = A(\theta)x(t, \theta) + B(\theta)u(t, \theta) \]
\[ u(t, \theta) = y(t, \theta) - C(\theta)x(t, \theta) \]

Theorem. The system (*) has a 1-generalized inverse system given by
Systems of the form \( E \dot{x}(t) = Ax(t) + Bu(t), y(t) = Cx(t) \), with \( E \) singular arise naturally when these systems are formed from the interconnected subsystems. Any system can usually be considered as an interconnection of subsystems when the natural differential equations and algebraic constraints which describe the systems are first written, they involve the internal variables of the subsystems in a description of the form above, and usually occur with \( E \) as a singular matrix.

This occurs for example when considering electrical networks using the element currents and voltages as internal variables.

\[
\begin{bmatrix}
\dot{x}(t,e) \\
\dot{y}(t,e)
\end{bmatrix} = \begin{bmatrix}
E^T(-\theta + E -E^T(-\theta)E) & 0 \\
0 & E^T(-\theta + E -E^T(-\theta)E)
\end{bmatrix}\begin{bmatrix}
x(t,e) \\
y(t,e)
\end{bmatrix} + \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

for arbitrary \( z_1, z_2 \). Therefore a 1-generalized inverse system of (*) is given by the following:

\[
\begin{bmatrix}
\dot{x}(t,e) \\
\dot{y}(t,e)
\end{bmatrix} = \begin{bmatrix}
E^T(-\theta + E -E^T(-\theta)E) & 0 \\
0 & E^T(-\theta + E -E^T(-\theta)E)
\end{bmatrix}\begin{bmatrix}
x(t,e) \\
y(t,e)
\end{bmatrix} + \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

and the generalized inverse systems is given below:

\[
\begin{bmatrix}
\dot{x}(t,e) \\
\dot{y}(t,e)
\end{bmatrix} = \begin{bmatrix}
E^T(-\theta + E -E^T(-\theta)E) & 0 \\
0 & E^T(-\theta + E -E^T(-\theta)E)
\end{bmatrix}\begin{bmatrix}
x(t,e) \\
y(t,e)
\end{bmatrix} + \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

\[
\begin{bmatrix}
\dot{x}(t,e) \\
\dot{y}(t,e)
\end{bmatrix} = \begin{bmatrix}
E^T(-\theta + E -E^T(-\theta)E) & 0 \\
0 & E^T(-\theta + E -E^T(-\theta)E)
\end{bmatrix}\begin{bmatrix}
x(t,e) \\
y(t,e)
\end{bmatrix} + \begin{bmatrix} 0 \\
0
\end{bmatrix}
\]

for arbitrary \( z_1, z_2 \).
In cases where \( E \) is a non-singular matrix it may be of interest to analyze the behavior of a system as above to that of a simplified or idealized model of it in which \( E \) is singular, i.e. this is characteristic of investigations of singularly perturbed systems. Discrete-time versions of the above equations also occur often. Thus systems of the above form have been studied considerably. The use of the Laplace transform with initial conditions given at \( t=0^- \) is made to obtain solutions of such systems including the impulsive parts of such solutions. Early work in this area was done by Doetsch [1], Erdelyi [2], and Liverman [4]. The detailed structure of the finite frequency or exponential modes of such equations is known.

When the system equation \( ES = Ax + Bu \) describes a system formed at \( t=0 \) as a result of impulse solutions.

\[ Y(t,B) = C(0)x(t,B) + D(B)u(t,B) \]

The systems described above are often used to consider inconsistent initial conditions. The matrix \( E \) is singular because of idealizations or approximations made in some other singularly perturbed system that could have initial conditions unconstrained by \( E \).

\[ ES = Ax + Bu \]

Also in switched composite systems it may be of interest to find out how subsystem properties are reflected into those of the composite and one needs to have an understanding of the consequences of interconnection of subsystems.

The approach used in this work is to obtain various generalized inverses of the original system in order to compute feed-back control laws and be able to consider the properties of stabilizability, controllability, and observability of the generalized inverse system of the original system. This approach is in contrast to the approach of allowing system equivalence of the original system which preserves important properties related to system behavior at infinity. In the study of the form \( E(\epsilon)x(t,\epsilon) = A(\epsilon)x(t,\epsilon) + B(\epsilon)u(t,\epsilon) \), \( y(t,\epsilon) = C(\epsilon)x(t,\epsilon) + D(\epsilon)u(t,\epsilon) \), \( t \geq 0 \), bifurcation points which arise in the \( \epsilon \)-parameter space lead into singular systems and impulsive modes of the solution \( x(t,\epsilon) \).

Electrical networks are natural candidates for analysis as composite systems since the are formed by interconnecting in various configurations a few standard components whose detailed behavior is known. The above approach is also robust in the sense that some nonlinear dynamical systems may be treated using inverse systems and may be implemented by use of \( E \)-structure inverse systems.

Example. Consider the time-invariant multi-parameter system of \( r \) first order coupled linear differential equations

\[ E(\epsilon)x(t,\epsilon) = A(\epsilon)x(t,\epsilon) + B(\epsilon)u(t,\epsilon) \]

\[ y(t,\epsilon) = C(\epsilon)x(t,\epsilon) + D(\epsilon)u(t,\epsilon), \quad t \geq 0 \]

where \( E(\epsilon)x(0^-,\epsilon) \) are known and are not necessarily \( 0 \), and where \( x(t,\epsilon) \) is an \( r \)-vector of internal variables, \( u(t,\epsilon) \) is an \( r \)-vector of control inputs or forcing functions, and \( y(t,\epsilon) \) is a \( p \)-vector of outputs. \( \phi^\epsilon(0,\epsilon_1,\epsilon_2,\ldots,\epsilon_n) \) is a multiparameter. Consider the case

\[ E(\epsilon) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \quad A(\epsilon) = \begin{bmatrix} 0 & 0 \\ 1 & 0 \end{bmatrix}, \quad B(\epsilon) = \begin{bmatrix} 0 \\ 1 \end{bmatrix}, \quad C(\epsilon) = (1, 0), \quad D(\epsilon) = (1), \]

\( E(0^-,\epsilon) = 0 \) for some parameter \( \epsilon \). The systems free-response has no exponential motions but does have an impulsive motion. Let \( u(t,\epsilon) = 0 \), then

\[ y(t,\epsilon) = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t,\epsilon) + D(\epsilon)u(t,\epsilon), \quad t \geq 0 \]

implies that:

\[ \begin{bmatrix} x_1(t,\epsilon) \\ x_2(t,\epsilon) \end{bmatrix} = \begin{bmatrix} x_1(t,\epsilon) + D(\epsilon)u(t,\epsilon) \\ x_2(t,\epsilon) \end{bmatrix}, \quad t \geq 0 \]

and this pair of equations has the impulsive solution

\[ \begin{bmatrix} x_1(t,\epsilon) \\ x_2(t,\epsilon) \end{bmatrix} = \begin{bmatrix} \phi_1(0^-,\epsilon) \phi_2(0^-,\epsilon) \\ 0 \end{bmatrix}(t,\epsilon), \quad t \geq 0 \]

since the steps that follow from \( x_2(t,\epsilon) \)

changing from its value \( x_2(0^-,\epsilon) \) to the value \( 0 \)

which is required by \( x_2(t,\epsilon) = 0 \) for \( t \geq 0 \)

is differentiated according to \( x_1(t,\epsilon) = \phi_1(0^-,\epsilon) \phi_2(0^-,\epsilon) \)

and appears as an impulse in \( x_1(t,\epsilon) \). This may also be seen by making use of the Laplace transform of the equation

\[ \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix} x(t,\epsilon) + D(\epsilon)u(t,\epsilon), \quad t \geq 0 \]

for a fixed \( \epsilon \) and allowing \( E(0^-,\epsilon) \) to not necessarily equal to zero.

REFERENCES

