Polar Signal Detection: Multi-Carrier Waveform Design for Improved Radar Detection Performance

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Abstract—Inspired by the success of multi-carrier waveforms such as Orthogonal Frequency Division Multiplexing (OFDM) in radio communication, multi-carrier based radar waveform designs have gained strong interest recently. Reasons for this include their resistance to multipath fading, ability to overcome the limitations of a congested frequency spectrum, ability to exploit frequency diversity gains, and potential to perform radar and communication functions simultaneously within the same hardware using the same waveform. This paper provides recent research on multi-carrier waveform design for joint radar and communication systems. First, we employ OFDM to modulate Multi-Frequency Complementary Phase Coded (MCPC) sequences in order to improve radar range resolution performance. Next, it is shown that even though these MCPC sequences produce periodic autocorrelation sidelobes with deep nulls, their overall sidelobe level is higher than that of other traditional pulse-compression radar waveforms such as Linear Frequency Modulated (LFM) waveforms. A method, termed Polar Signal Detection, is introduced as a means of overcoming these large autocorrelation sidelobes without sacrificing range measurement resolution.

I. INTRODUCTION

MULTICARRIER based radar waveform designs have recently gained strong interest due in part to the success of multi-carrier waveforms such as Orthogonal Frequency Division Modulation (OFDM) in radio communication technologies. From a radar-centric point of view, multi-carrier waveforms can be used to improve detection and measurement performance [1]–[3]. Furthermore, the use of such waveforms improves resistance to multipath fading [4], [5], the ability to overcome the limitations of a congested frequency spectrum [6], the ability to exploit frequency diversity gains stemming from the fact that target scattering centers inherently resonate differently at different frequencies [7], and the potential to perform radar and communication functions simultaneously within the same hardware using the same waveform [8]–[10]. In comparison to traditional single carrier systems, a multi-carrier radar provides more degrees of freedom in waveform synthesis.

The range (or delay) resolution of a radar system is inversely proportional to the transmitted signal bandwidth. Realizing that improving range resolution usually entails employing a shorter bit duration in a digital phase modulated system or a wider frequency spread in an analog frequency modulated system, Levanon [11]–[13], and Levanon and Mozeson [14], [15] examined the use of multi-carrier communication technologies and MCPC sequences to increase the range resolution of radar systems. MCPC signals are sequences of signals, each different from one another, that form a complementary set. A complementary set is defined as cyclically shifted versions of a phase coded sequence having an ideal periodic autocorrelation function (ACF) [16]. Two such codes (which form the basis of MCPC signals) are the polyphase P3 and P4 code sequences. The phase terms of an M length P3 code are constructed as

\[ \phi_m = \frac{\pi}{M} (m - 1)^2 - \pi (m - 1) \]  

(1)
for \( m = 1, 2, \ldots, M \), and the phase terms for an \( M \) length P4 code are constructed as

\[
\phi_m = \begin{cases} 
\frac{\pi}{M} (m-1)^2 & m \text{ is even} \\
\frac{\pi}{M} (m-1) & m \text{ is odd}
\end{cases} \tag{2}
\]

for \( m = 1, 2, \ldots, M \). The MCPC signal is an \( M \times M \) matrix of cyclically shifted versions of either \( M \) length P3 or \( M \) length P4 codes. It is possible to stack MCPC sequences resulting in a \( M \times N \) sequence where \( N \) is an integer multiple of \( M \). Such a stacked sequence also forms a complementary set. TABLE I provides an example of a P3 based \( 5 \times 5 \) MCPC signal.

Using a traditional OFDM signaling scheme, the \( M \) sequences are transmitted on \( N \) subcarriers separated in frequency by

\[
\Delta f = \frac{BW}{N} = 1/t_b, \tag{3}
\]

where \( BW \) is the bandwidth of the system and \( t_b \) is the time duration of each bit in the sequence. The total duration of the transmitted pulse is then \( MT_b \). The transmitted pulse is mathematically defined as

\[
X(t) = \sum_{n=1}^{N} \exp \left( j2\pi t \Delta f \left( \frac{N+1}{2} - n \right) \right) \cdot \sum_{m=1}^{M} u_{n,m} (t - (m-1) t_b), \tag{4}
\]

for \( 0 \leq t \leq MT_b \) (otherwise \( X(t) = 0 \)) where

\[
u_{n,m} = \begin{cases} 
\exp(j\phi_{n,m}) & 0 \leq t \leq t_b \\
0 & \text{otherwise}
\end{cases} \tag{5}
\]

and \( \phi_{n,m} \) is the \( m^{th} \) phase element of the \( n^{th} \) sequence (i.e. the \( m^{th} \) column and \( n^{th} \) row of TABLE I). It is observed that this OFDM modulation scheme can be efficiently accomplished with the Inverse Fast Fourier Transform (IFFT) operation and that, upon reception, the transmitted sequence can be demodulated through application of the Fast Fourier Transform (FFT).

A measure of performance of a radar signal is its autocorrelation sidelobe response. A typical approach to reduce sidelobe levels is to apply a weighting (i.e. a Hamming window) to the transmitted pulse. This weighting results in reduced sidelobes with the consequence of a wider center response which causes a reduction in range measurement accuracy. Fig. 1 compares the autocorrelation response of an MCPC P3 signal, an LFM signal that has a total bandwidth to provide the same range measurement resolution as the P3 signal, and the same LFM signal with a Hamming window applied. It is clear that the windowed signal produces much lower sidelobes at the cost of a wider center response and that the sidelobe levels of the P3 signal are much larger than the others.

### III. POLAR SIGNAL DETECTION

As described by Levanon, the benefit of MCPC based codes is that they produce ideal (null) periodic autocorrelation sidelobs as shown in Fig. 1. This normalized ACF indeed demonstrates that the ACF produces deep nulls at integer multiples of \( t_b \) as expected. However, it is clear that this particular sequence (and all P3 and P4 based MCPC signals in general) has relatively high overall sidelobe levels. These large sidelobes can cause false detections or even missed detections due to sidelobe masking of weaker target returns.

#### A. The Beta Detector

P3 and P4 based MCPC codes are orthogonal. That is, an \( M \times M \) MCPC code multiplied by its conjugate transpose results in the \( M \times M \) identity matrix \( I \). This property will be exploited to overcome the inherent problem of large MCPC ACF sidelobes.

As a radar receives a signal it performs an estimation of the underlying bits by performing an IFFT operation on the signal (readers are referred to [17] for more information on OFDM detection and technologies). Assuming a sampling period of \( t_s \), a total of \( MT_b/t_s \) estimates will be produced. These estimated bits are reconstructed into a \( M \times M \) MCPC matrices which are subsequently multiplied by the conjugate transpose of the transmitted MCPC sequence resulting in

\[
\hat{\xi}_n = \hat{U}_n U^*, \quad n = 1, 2, \ldots, MT_b/t_s. \tag{6}
\]

Remembering that MCPC sequences are orthogonal, \( \hat{\xi}_n \) in (6) will equal an \( M \times M \) identity matrix when the estimated MCPC matrix \( \hat{U}_n \) equals the transmitted MCPC matrix \( U \).

---

**TABLE I**

**PHASES IN DEGREES OF THE ELEMENTS OF A P3 BASED 5X5 MCPC SIGNAL**

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A vector of measurements $R(n)$ are taken on all $n$ $\hat{\xi}$ values such that

$$R(n) = \frac{\text{tr} \left( \xi_n \right)^2}{M \cdot \text{sum} \left( \xi_n^2 \right) + \text{tr} \left( \xi_n^2 \right)^2}$$

$$n = 1, 2, \ldots, M_{tb}/t_s$$  \hspace{1cm} (7)

where $\text{tr} \left( \cdot \right)$ is the trace operation, and $\text{sum} \left( \cdot \right)$ is the sum of all elements in the matrix. When the estimated MCPC matrix equals the transmitted MCPC matrix, $R$ in (7) will equal 0.5.

It is necessary to derive the statistical distribution of (7) to establish an appropriate detection threshold. The following derivations will be concerned with the null hypothesis (the case where no signal is present). For the null hypothesis, each element of $\xi_n$ is normally distributed as $N \left( 0, \sigma_x^2 \right)$. Considering the numerator of (7), it is obvious that the trace operation of the elements of $\xi_n$ will also be normally distributed as $N \left( 0, M\sigma_x^2 \right)$. The absolute square of the trace then becomes gamma distributed as $\Gamma \left( 1, 2M\sigma_x^2 \right)$.

Now considering the first term in the denominator of (7), the sum of the absolute squares of the elements of $\xi_n$ are gamma distributed as $\Gamma \left( M^2, 2\sigma_x^2 \right)$. Multiplying this gamma distribution by $M$ results in the gamma distribution $\Gamma \left( M^2, 2M\sigma_x^2 \right)$. The second term in the denominator of (7) is identical to the numerator. The resulting distribution takes the form

$$\frac{\Gamma \left( 1, 2M\sigma_x^2 \right)}{\Gamma \left( M^2, 2M\sigma_x^2 \right) + \Gamma \left( 1, 2M\sigma_x^2 \right)} = \beta \left( 1, M^2 \right),$$  \hspace{1cm} (8)

where $\beta$ is the beta distribution. Now the desired the detection threshold $T_{\beta}$ for a desired $P_{FA}$ can be determined by solving

$$1 - P_{FA} = \frac{1}{B \left( 1, M^2 \right)} \int_0^{T_{\beta}} \left( 1 - x \right)^{M^2 - 1} dx$$

for $T_{\beta}$ ($B \left( \cdot \right)$ is the beta function). An exact solution for the threshold does not exist, but can be numerically approximated. This detection scheme will henceforth be referred to as the beta detector (BD).

B. Cell Averaging Constant False Alarm Rate Detectors

To benchmark the performance of the detector proposed in this paper, various detection schemes will be examined including the cell averaging constant false alarm rate (CA-CFAR) detector. CA-CFAR detectors assume that neighboring resolution cells to the cell under test (CUT) share the same statistics as the CUT. Furthermore, CA-CFAR detectors rely on neighboring cells not having any targets. CA-CFAR detectors use a number of reference cells on either side of the CUT to estimate the interference power at the CUT. Cells directly adjacent to the CUT called guard cells are used to avoid corrupting the interference estimate with power from the CUT itself.

Additional CA-CFAR structures are also used. One such is termed Least-Of CA-CFAR (LO-CA-CFAR). It works by comparing the power estimates in the leading and lagging reference cells and using the smaller of the two in the derivation of the threshold. The purpose of the LO-CA-CFAR, is to prevent target masking. Target masking can occur when targets are present within the reference cells causing the interference estimate to be too large. The LO-CA-CFAR, attempts to circumvent this by only using the least-of the interference estimates. However, this methodology will fail when targets are present within both the leading and lagging reference cells.

Another CA-CFAR scheme is called Greatest-Of CA-CFAR (GO-CA-CFAR) The GO-CA-CFAR, compares the interference estimates from the leading and lagging reference cells and chooses the greatest of the two. The purpose of this type of detector is to prevent false alarms caused by "spikes" in the interference power caused by clutter ridges or any other strong source of interference. However, this scheme will exacerbate target masking issues.

Yet another structure is called Ordered-Statistic CA-CFAR (OS-CA-CFAR). This structure works by ordering the power measurements in the reference cells from smallest to largest and taking assigning the $K^{th}$ values as the estimate for the interference. Unlike the previously described CFAR methods, in OS-CA-CFAR the interference is estimated from only one sample instead of an average of samples. However, the threshold is still dependent on all of the samples as the $K^{th}$ largest value in the ordered statistic is dependent on all of the samples. The purpose of this type of detector is to help prevent target masking.

After the interference at the CUT has been estimated, the next step is to assign an appropriate threshold to maintain a desired $P_{FA}$. This is accomplished through

$$T = \sqrt{\alpha \hat{I}},$$

where $\alpha$ is a threshold multiplier and $\hat{I}$ is the interference estimate. The value of $\alpha$ in (10) is dependent on the number of training cells, the desired $P_{FA}$ and the CFAR type. For the basic CA-CFAR detector, $\alpha$ is calculated as [18]

$$\alpha_{CA} = N \left( P_{FA}^{-1} - 1 \right),$$

where $N$ is the number of training cells. Similar closed form solutions for the threshold multiplier for the other CFAR detectors do not exist and must be solved for iteratively. For the LO-CA-CFAR detector, the equation to solve is [19]

$$P_{FA} = \left( 2 + \alpha_{LO} \frac{N}{2} \right)^{-N/2} \sum_{k=0}^{N/2-1} \binom{N/2}{k+1} \left( 2 + \alpha_{LO} \frac{N}{2} \right)^{-k},$$

for the GO-CA-CFAR detector, the equation to solve is [19]

$$P_{FA} = \left( 1 + \frac{\alpha_{GO}}{N/2} \right)^{-N/2} - \left( 2 + \frac{\alpha_{GO}}{N/2} \right)^{-N/2} \times \sum_{k=0}^{N/2-1} \binom{N/2}{k+1} \left( 2 + \frac{\alpha_{GO}}{N/2} \right)^{-k},$$

and for the OS-CA-CFAR detector, the equation to solve is [18]

$$P_{FA} = \frac{N! \left( \alpha_{OS} + N - k \right)!}{\left( N - k \right)! \left( \alpha_{OS} + N \right)!},$$

where $\alpha_{OS}$ is an integer. Readers are referred to [18] for a more detailed analysis on CFAR techniques.
C. Detection Example

An example will now be given to further the discussion. Assume that a radar is employing $6 \times 6$ P3 based MCPC waveforms. Also, assume there are 7 targets located within the pulse response at $t/t_{b} = -4.0644, t/t_{b} = -2.4797, t/t_{b} = -1.2581, t/t_{b} = 0.0, t/t_{b} = 0.2806, t/t_{b} = 1.3834,$ and $t/t_{b} = 5.5905.$ Furthermore, assume that the signal to noise ratio at the receiver is 10 dB for the target at $t/t_{b} = 0$ and 0 dB for the remaining targets.

For this example, each CFAR detector will be designed to achieve a $P_{FA}$ of $10^{-5}$ and will have leading and lagging training samples each spanning $t/t_{b} = 1$ and leading and lagging guard samples each spanning $t/t_{b} = 0.333.$ As is typical [20], the value of $K$ used in the OS-CA-CFAR detector is set to three quarters of the total number of training samples.

Fig. 2 shows the matched filter response, the Neyman-Pearson detection threshold [21] (labeled $T_{MF}$), as well as the various CFAR detection thresholds. The constant threshold $T_{MF}$ produces an excessive number of false alarms due to the large range sidelobe response, while the various CFAR detection thresholds experience fewer false alarms. Fig. 3 shows the same scenario for the BD where only the target at $t/t_{b} = 0$ is detected. In this example no detector was successful in detecting all 7 targets.

D. Derivation of the Polar Signal Detection Method

A new detection scheme called the Polar Signal Detection (PSD) method will now be provided as a means of improving upon the previously presented detection performances. Instead of examining beta detection statistics and matched filter detection statistics individually with respect to range, they will now be considered jointly. Fig. 4 shows the same response data previously presented but now plotted as the beta statistic versus the matched filter response. The various shapes within the figure represent the individual target’s effect on the overall joint response. Qualitatively, each target is easily discernible.

The next step is to derive a new threshold parameter that will quantitatively detect the targets.

A polar representation of the data presented in Fig. 4 with an origin at $(0, 0)$ is now constructed and shown in Fig. 5. In this figure, as $\theta$ approaches 0, the $\rho$ values are more heavily influenced by the beta response and as $\theta$ approaches $\pi/2$, $\rho$ is more heavily influenced by the matched filter response. The tangent of $\theta$ equals the ratio of the MF response to the beta response.

Similar to the CA-CFAR approach, an interference estimate must be calculated from the available polar data. The approach taken is to use all samples less than $\theta_{I,\text{min}}$ and greater than $\theta_{I,\text{max}}$ for interference estimation. Through experimentation, it has been found that $\theta_{I,\text{min}} = 0.4$ and $\theta_{I,\text{max}} = 1.5$ are acceptable values to use. The statistical distribution of samples

![Matched Filter Response](image1)

![Beta Response](image2)

![Joint Response](image3)
below θ_{1,\text{min}} = 0.4 will resemble the same beta distribution as derived in (8), and it can be shown that the distribution of samples greater than θ_{1,\text{max}} = 1.5 will follow a gamma distribution.

A gamma distribution takes the form
\[ p(x|a,b) = \text{Gamma}(x; a, b) = \frac{x^{a-1}}{\Gamma(a) b^a} \exp\left(-\frac{x}{b}\right), \] (15)
where \( \Gamma(\cdot) \) is the gamma function. The parameters \( a \) and \( b \) can be found using a maximum likelihood approach. Following along with [22], assuming there are \( n \) samples available for estimation, the log-likelihood equation is
\[
\log(p(D|a,b)) = n(a - 1) \log(x) - n \log(\Gamma(a)) - na \log(b) - \frac{n\bar{x}}{b}, \] (16)
where the over-bar signifies the mean value. The maximum for \( b \) is
\[ \hat{b} = \frac{\bar{x}}{a}. \] (17)
Substituting (17) into (16) results in
\[
\log(p(D|a,\hat{b})) = n(a - 1) \log(x) - n \log(\Gamma(a)) - na \log(\bar{x}) + na \log(\log(x) + n(1 + \log(a)) - n, \] (18)
which needs to be iteratively maximized to find \( a \). The maximum is at
\[ 0 = n \log(x) - n \Psi(a) - n \log(\bar{x}) + n(1 + \log(a)) - n, \] (19)
where
\[ \Psi(a) = \log(x) - \log(\bar{x}) + \log(a). \] (20)
Now, (19) is iterated by setting \( a_0 \) to the current estimate of \( a \). A solution that converges in about 4 iterations is found by using a generalized Newton method [23], where an approximation of the form
\[ \log(p(D|a,b)) \approx c_0 + c_1 a + c_2 \log(a), \] (21)
leads to an update of \( a \) in the form of
\[ \frac{1}{a_{\text{new}}} = \frac{1}{a} + \frac{\log(x) - \log(\bar{x}) + \log(a) - \Psi(a)}{a^2(1/a - \Psi'(a))}. \] (22)
Now, the detection threshold for a desired \( P_{FA} \) can be found through the inverse cumulative gamma distribution. That is, by solving
\[ 1 - P_{FA} = \frac{1}{b^a \Gamma(a)} \int_0^{\theta} x^{a-1} e^{-x/b} dx \] (23)
for \( T_\Gamma \). Like the beta distribution, an exact solution for the threshold does not exist, but can be solved for numerically.

The question now, is how to transition from the threshold as found through the beta distribution to the threshold as found through the gamma distribution. The approach taken here is to use the quadratic Bèzier curve which is defined as
\[ B(t) = P_0 (1 - t)^2 + 2P_1 t (1 - t) + P_2 t^2, \quad 0 \leq t \leq 1. \] (24)
The path of a Bèzier curve begins at point \( B(0) = P_0 \), travels towards \( P_1 \) (without ever reaching \( P_1 \)), then turns and ends at \( B(1) = P_2 \).

The Bèzier points used to generate a threshold for the PSD method are \( P_0 (0, T_\beta) \), \( P_1 (\theta_{\text{\beta,max}}, T_\beta) \), and \( P_2 (\theta_{\text{\beta,max}}, T_\Gamma) \), where \( \theta_{\text{\beta,max}} \) is the \( \theta \) location of the sample with the largest value of \( \rho \). For the example case, the three Bèzier points are \( P_0 (0, 0.2501) \), \( P_1 (1.931, 0.2501) \), \( P_2 (1.4, 0.4013) \) each of which are marked in Fig. 5. In examining Fig. 5, it is seen that the Bèzier threshold follows the same basic curve as the underlying interference samples. For illustrative purposes, the Bèzier threshold is converted back into Cartesian space and is plotted in Fig. 4. Fig. 4 and Fig. 5 clearly demonstrate that six of the seven targets are detected using PSD.

### E. Simulation

Now the PSD method will be evaluated in terms of its performance as compared to the previously mentioned CFAR detectors. For this test, a \( 6 \times 6 \) P3 based MCPC sequence will be used to detect 7 targets with a desired \( P_{FA} \) of \( 10^{-5} \). One target is always placed at \( t/t_b = 0.0 \) while the others are randomly placed between \( t/t_b = -6.0 \) and \( t/t_b = 6.0 \) inclusive. The return from the target at \( t/t_b = 0.0 \) always has a SNR of 10.0 dB. The independent test variable is the echo power from the targets not at \( t/t_b = 0.0 \). For each test, the return power from these targets will all be equal and varied from \(-10.0 \) dB to \( 6.0 \) dB.

Fig. 6 shows the results of this test. The curves in this figure are the probabilities of detection versus the received power from the targets not located at \( t/t_b = 0.0 \). In this case, the probability of detection is equal to the total number of target detections divided by the total number of targets. It is clear from Fig. 6 that the detection performance of the PSD detector is superior to the CA-CFAR based detectors.

Fig. 7 provides the \( P_{FA} \) in terms of the tested SNR. It is noted that none of the detectors meet the desired \( P_{FA} \) of \( 10^{-5} \). Nevertheless, the \( P_{FA} \) for the various detectors are relatively constant across all tested SNRs. It is observed that, out of all
term the Polar Signal Detection, was introduced as a means of overcoming these large autocorrelation sidelobes. Through simulation, it was shown that this new detector provides superior detection performance as compared to CA-CFAR, LO-CA-CFAR, GO-CA-CFAR, and SO-CA-CFAR detectors in multi-target environments.

**REFERENCES**


