Modal Analysis of One-Dimensional Electromagnetic Metamaterial Grounded Slab

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Abstract—Electromagnetic Metamaterials (MTMs) are artificial materials with novel electromagnetic properties not available in nature. MTMs have the potential to facilitate significant improvement on performance of low-profile (i.e. microstrip) and conformal antennas, including reduction of antenna size and antenna coupling. In this paper, we develop analytic expressions and corresponding tractable approximations for impedance and dispersion relations for one-dimensional MTMs applicable to a wide frequency range. Our technique obtains properties relevant for the application of MTMs to antennas without the need to derive the effective medium parameters first. We apply this technique to investigate surface wave modes supported by a single-layer and double-layer one-dimensional electromagnetic crystal on a ground plane.

We use modal analysis for one-dimensional stratified periodic MTM media. The modal formalism is directly applicable to the surface wave problem and bypasses difficulties associated with defining average constitutive parameters that are valid only in the quasi-static region. We use transmission-line theory to solve the propagation problem in one direction, while modal functions are determined for the transverse plane. The stratified material is decomposed into unit cells. We apply Floquet's theorem and the chain matrix method to determine the characteristic impedance and the dispersion relation. We develop algebraic approximations to trigonometric functions in order to obtain approximate expressions in various frequency regimes. These approximations can be used to determine reactive impedance (stopgap) regions, surface-wave modes Greens functions (Antenna patterns), and resonance conditions for microstrip and conformal antennas.

We apply this technique to the open slab configuration using transverse resonance. This work is an initial stage in CERDEC's effort to develop an approach to tailor MTM properties to meet military application-driven antenna requirements.

I. INTRODUCTION

Research into electromagnetic Metamaterials (MTMs) represent one of the most active research areas today as demonstrated by numerous research articles and several recent books [1]–[4]. MTMs have significant potential applications for electromagnetic technologies. One of the most promising MTM applications is development of improved antennas for military applications. Advantages expected from the use of MTMs include reduction in antenna size, improvements in antenna performance, and improvement in advanced antenna systems (conformal antenna arrays, etc.).

MTMs are artificial electromagnetic structures with unusual properties not readily available in nature [2]. MTMs represent a very active research area due to the variety of unusual effects that can be obtained using metamaterials. Typically, MTM are either periodic structures or random, e.g. percolative structures. Periodic MTMs have one or more types of elements inserted into a host material and forming a periodic structure. Examples of inclusions include, balls, rods, or split-ring resonators (SRRs). Inclusions could also be fabricated as lumped circuit elements inserted into transmission line structures. MTMs include one-dimensional and multi-dimensional photonic crystals [5].

By analogy with natural materials, behavior of periodic MTMs can be separated into the following two limiting regimes depending on the wavelength:

- **Effectively homogeneous.** The concept of the homogeneous structure implies the ability to average the microscopic field to define a macroscopic field in order to define effective constitutive parameters (material constants).
- **Photonic Band Gap (PBG).** PBG are periodic structures operating at frequencies where the lattice period is of the order of multiple of half a wavelength. Stop bands are created where wave propagation is not possible in the same manner as occurs for electron wavefunctions in solid crystals.

In this paper we seek to understand MTM behavior over a frequency range that spans both these limits. In addition, we would like to determine tractable expressions that can be directly useful in the antenna design work. Therefore, we analyze a one-dimensional electromagnetic crystal as a test case to investigate useful approximations. This configuration is important and can be analyzed exactly. In view of the difficulties associated with defining effective homogeneous parameters over a wide frequency range, we work directly with the impedance and wavenumber in the modal decomposition. Finally, we determine the surface wave modes of an MTM slab over a ground plane. This problem serves as an illustration and a test for the usefulness of the approximations.

II. MODEL DESCRIPTION

A. Material Configuration

We consider a one-dimensional MTM that consists of a periodic mutilayer structure. In a general case, there are \( m \) slabs of arbitrary thickness repeating periodically. Typically, one of the layers would be the “host” material while the remaining layers would be the “embedded.” The host layer is expected to be considerably thicker compared to the embedded layers. While we formulate the basic theory in terms of an arbitrary multilayer material, we will focus on two important cases:

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Fig. 1. Illustration of single-layer media a single embedded electric or magnetic layer.

- **Single-layer.** A single-layer MTM is a material with a periodic inclusion comprising a single embedded dielectric or magnetic layer.

- **Double-layer.** A double-layer MTM is material with a periodic inclusion comprising comprising layers of identical width. One layer is a dielectric layer while the other is a magnetic layer.

Figure 1 illustrates the single layer MTM, and Figure 2 illustrates the double-layer MTM.

**B. Partition into Unit Cells**

Figure 1 and 2 shows the partition of the periodic material into unit cells. We choose the unit cell boundary to make the unit cell as symmetric as possible. The unit cell boundary is chosen in the center of the layer of the host material. Thus, the first layer and the last layer of the unit cell are identical. The unit cell is symmetrical if the embedded layer structure is symmetrical.

Each unit cell has \( m + 1 \) slabs and \( m + 2 \) interfaces. Each layer has a thickness \( l_i \), permeability \( \mu_i \) and permittivity \( \epsilon_i \) where \( i \in \{1..m+1\} \). Due to the symmetrizing choice of the cell, the parameters of the first and last layers are the same:

\[
\epsilon_1 = \epsilon_{m+1}, \quad \mu_1 = \mu_{m+1}, \quad l_1 = l_{m+1}. \quad (1)
\]

The total length of the unit cell is \( l = \sum_{i=1}^{m+1} l_i \). We associate propagation velocity \( v_i \) and wavelength \( \lambda_i \) with each layer:

\[
v_i = \frac{1}{\sqrt{\mu_i \epsilon_i}}, \quad \lambda_i = \frac{v_i}{f}, \quad (2)
\]

where \( f \) is the frequency.

To simplify the notation, we adopt normalized system of units where \( \epsilon_0 = 1, \mu_0 = 1, \) and \( l = 1 \). In this system of units \( k_0 = \omega \).

**III. GUIDED WAVE REPRESENTATION**

**A. Modal Decomposition**

In the general case of a media stratified in one dimension (along the \( z \) axis) and homogeneous along the transverse direction (\( x \) and \( y \) axis), the electromagnetic fields can be decomposed into TE, TM, and TEM modes [6]. The EM field equations for each mode correspond to a transmission line equations in the \( z \) direction with the following parameters [7]

\[
\kappa_i = \sqrt{\omega^2 \mu_i \epsilon_i - k_l^2} = \sqrt{k_0^2 n_i^2 - k_l^2}, \quad (3)
\]

\[
Z_i = \begin{cases} \frac{\omega \mu_i}{\kappa_i} & \text{TE mode,} \\ \frac{\omega \epsilon_i}{\kappa_i} & \text{TM mode.} \end{cases} \quad (4)
\]

In the most general case, the allowable values of \( \kappa_i \) are determined by the boundary conditions in the \( x \)-\( y \) plane. For unbounded media, \( \kappa_i \) has a continuous spectrum. For a propagating plane wave, \( 0 \leq \kappa_i \leq k_0 n_i \) and \( \kappa_i = k_0 n_i \cos(\theta_i) \) where \( \theta_i \) is the angle between the direction of propagation and the \( z \)-axis. The TEM mode \( (E_z = 0, H_z = 0) \), which for plane waves corresponds to a case of normal incidence, has transmission line parameters given by:

\[
k_l = 0, \quad \kappa_i = \omega \sqrt{\mu_i \epsilon_i} = k_0 n_i, \quad Z = \frac{\mu_i}{\epsilon_i}. \quad (5)
\]

**B. Unit Cell Transmission Line Model**

We can obtain an equivalent transmission line model for each unit cell as shown in Figure 3 and Figure 4. Each transmission line section can be represented by an
ABCD matrix $\mathbf{M}_i$ (chain matrix) as follows [8], [9]: 
\[
\mathbf{M}_i = \begin{bmatrix} A_i & B_i \\ C_i & D_i \end{bmatrix} = \begin{bmatrix} \cos (\kappa_i l_i) & jZ_i \sin (\kappa_i l_i) \\ jY_i \sin (\kappa_i l_i) & \cos (\kappa_i l_i) \end{bmatrix}
\]
(6)
where $Y_i = \frac{1}{Z_i}$. 

The total ABCD matrix $\mathbf{M}$ is the product of the matrices representing the unit-cell layers. Taking into account that the last unit cell layer is the same as the first, the matrix is 
\[
\mathbf{M} = \begin{bmatrix} A & B \\ C & D \end{bmatrix} = \mathbf{M}_1 \left( \prod_{i=2}^{m} \mathbf{M}_i \right) \mathbf{M}_1.
\]
(7)

The product of the matrices inside the parenthesis represents the effect of the embedded layers, while the matrices outside the parenthesis represent the host material. If we call the product inside the parenthesis $\mathbf{M}_\text{em}$, then we can multiply the matrices out explicitly as follows:
\[
A = A_\text{em} \cos^2 (\kappa_1 l_1) - D_\text{em} \sin^2 (\kappa_1 l_1) + j \left( C_\text{em} Z_1 + B_\text{em} Y_1 \right) \sin (\kappa_1 l_1) \cos (\kappa_1 l_1)
\]
(8a)
\[
B = B_\text{em} \cos^2 (\kappa_1 l_1) - C_\text{em} Z_1^2 \sin^2 (\kappa_1 l_1) + j Z_1 (A_\text{em} + D_\text{em}) \sin (\kappa_1 l_1) \cos (\kappa_1 l_1)
\]
(8b)
\[
C = C_\text{em} \cos^2 (\kappa_1 l_1) - B_\text{em} Y_1^2 \sin^2 (\kappa_1 l_1) + j Y_1 (A_\text{em} + D_\text{em}) \sin (\kappa_1 l_1) \cos (\kappa_1 l_1)
\]
(8c)
\[
D = D_\text{em} \cos^2 (\kappa_1 l_1) - A_\text{em} \sin^2 (\kappa_1 l_1) + j \left( C_\text{em} Z_1 + B_\text{em} Y_1 \right) \sin (\kappa_1 l_1) \cos (\kappa_1 l_1)
\]
(8d)

For a symmetrical configuration these expressions simplify because $A_\text{em} = D_\text{em}$.

C. Exact Solution for Periodic Structure

The exact solution, in accordance with Floquet’s theorem, for the impedance and the wavenumber for infinite periodic structure is well-known [10]. It consists of Bloch waves, where
\[
k_B l = \pm \cos^{-1} \left( \frac{A + D}{2} \right)
\]
(9)

The solution corresponds to waves propagating in +z and -z directions. It is necessary to select the values corresponding to the Riemann sheet corresponding to the physical solution. Similarly, we can solve for the characteristic impedance of the periodic network by taking advantage of the periodicity. However, for the asymmetrical unit cell the solution depends on the direction of propagation. So we need to distinguish between the characteristic impedance for waves propagating to the right $\hat{Z}_B$ and waves propagating to the left $\hat{Z}_B$. Using $BC = AD - 1$, we can get the form shown by Collin [10]:
\[
\hat{Z}_B = \frac{2B}{D - A + \sqrt{(D + A)^2 - 4}}
\]
(10a)
\[
\hat{Z}_B = \frac{2B}{D - A - \sqrt{(D + A)^2 - 4}}
\]
(10b)

For symmetrical network $A = D$, $B = -C$, Eqs. 9 and 10 simplify as follows:
\[
k_B l = \pm \cos^{-1} (A),
\]
(11a)
\[
\hat{Z}_B = \hat{Z}_B = Z_B = \sqrt{\frac{B}{C}}.
\]
(11b)

Equations 9 and 10 (or 11) provide an explicit closed-form full-wave solution for the fields at the unit cell boundaries. If needed, the fields inside the unit cell are determined by standard transmission line equations. In terms of the embedded structure parameters defined by Eq. 8, the wavenumber is:
\[
k_B l = \pm \cos^{-1} \left( \frac{A_\text{em} + D_\text{em}}{2} \right) \cos (2\kappa_1 l_1)
\]
\[
+ j \frac{1}{2} \left( C_\text{em} Z_1 + B_\text{em} Y_1 \right) \sin (2\kappa_1 l_1)
\]
(12)

For symmetrical unit cell the wavenumber is:
\[
k_B l = \pm \cos^{-1} \left[ A_\text{em} \cos (2\kappa_1 l_1)
\]
\[
+ j \frac{1}{2} \left( C_\text{em} Z_1 + B_\text{em} Y_1 \right) \sin (2\kappa_1 l_1) \right].
\]
(13)

The characteristic impedance for symmetrical network becomes:
\[
Z_B = \left[ B_\text{em} \cos^2 (\kappa_1 l_1) - C_\text{em} Z_1^2 \sin^2 (\kappa_1 l_1)
\]
\[
+ j Z_1 A_\text{em} \sin (2\kappa_1 l_1) \right]^\frac{1}{2}/
\]
\[
\left[ C_\text{em} \cos^2 (\kappa_1 l_1) - B_\text{em} Y_1^2 \sin^2 (\kappa_1 l_1)
\]
\[
+ j Y_1 A_\text{em} \sin (2\kappa_1 l_1) \right]^\frac{1}{2}
\]
(14)

D. Single-Layer Material

We evaluate the exact dispersion relation for a single-layer material:
\[
\cos (k_B l) = \pm \cos (\kappa_2 l_2) \cos (2\kappa_1 l_1)
\]
\[
- \frac{1}{2} \left( \frac{Z_1^2 + Z_2^2}{Z_1} \right) \sin (2\kappa_1 l_1) \sin (2\kappa_2 l_2),
\]
(15)
while characteristic (Bloch) impedance is:

\[
Z_B = \left[ Z_2 \sin (\kappa_2 l_2) \cos^2 (\kappa_1 l_1) - Y_2 \sin (\kappa_2 l_2) \times \right.
\]
\[
\left. \frac{Z_1^2 \sin^2 (\kappa_1 l_1) + Z_1 \sin (2\kappa_1 l_1) \cos (\kappa_2 l_2) }{[Y_2 \sin (\kappa_2 l_2) \cos^2 (\kappa_1 l_1) - Z_2 \sin (\kappa_1 l_1) \times \right. \right]
\]
\[
\left. Y_1^2 \sin^2 (\kappa_1 l_1) + Y_1 \sin (2\kappa_1 l_1) \cos (\kappa_2 l_2) \right]^{\frac{1}{2}}
\]

(16)

E. Band Structure

The solution in Eq. 9 has a band structure consisting of stop bands and pass bands. This type of band structure is typical of periodic structures [11]. If \(|A + D| \leq 2\) the propagation constant is real and the solution corresponds to a pass band. If \(|A + D| > 2\), the propagation constant is imaginary, the wave is evanescent in the z direction and the solution corresponds to a stop band.

For a symmetric unit cell \(A = D\) so the band boundary corresponds to condition \(|A| = 1\). For a reciprocal unit cell \(AD - BC = 1\) so that \(BC < 0\) in the pass band and \(BC > 0\) in the stop band. For a lossless case \(B\) and \(C\) are purely imaginary so the characteristic impedance for the symmetrical cell is real in the pass band and imaginary in the stop band.

For an asymmetric unit cell, the impedance is complex in the pass band. In the stop band, the numerator is purely real, so the characteristic impedance for the symmetrical cell is real in the pass band and imaginary in the stop band.

IV. APPROXIMATIONS

A. General Considerations

Although we have obtained an exact full-wave solution, this solution relates details of the MTM internal structure to externally “visible” parameters through trigonometric functions. In order to work with electromagnetic systems that include MTMs, we would like approximations that provide fairly simple relations among the external quantities, impedance, wavenumber, and frequency. Ideally, these relationships should be invertible.

B. Quasi-Static Approximation (QSA)

The QSA approximation applies to any layer under the following conditions:

- Wavelength is large relative to the slab thickness, \(\lambda_i \gg l_i\).
- Wave direction along the slab, \(k \approx k_0 n_i\).

In this case, we can expand the trigonometric functions in Taylor series. The expansion to second order yields:

\[
\cos (\kappa_1 l_1) \approx 1 - \frac{(\kappa_1 l_1)^2}{2} \quad j Z_1 \sin (\kappa_1 l_1) \approx j \omega l_i \mu_i
\]

(17)

In many cases, QSA applies to the host material, but may not hold for the embedded layers. However, where the QSA can be used for all layers, the dispersion relation for a single-layer material becomes:

\[
2 \cos (k_B l) \approx 2 + (\zeta l_2)^2 + (2\zeta l_1)^2 - \left( \frac{Z_1}{Z_2} + \frac{Z_2}{Z_1} \right) (2\zeta l_1)(\zeta l_2)
\]

(18)

The corresponding expressions for the impedance are:

\[
Z_B = \sqrt{\frac{Z_2 \zeta_2 - Y_2 \zeta_2 (Z_1 \zeta_1)^2 + 2 Z_1 \zeta_1}{Y_2 \zeta_2 - Z_2 \zeta_2 (Y_1 \zeta_1)^2 + 2 Y_1 \zeta_1}} [1 + \zeta_2^2]
\]

(19)

where \(\zeta_i = \kappa_i l_i\). For example, for the TEM mode this becomes

\[
2 \cos (k_B l) \approx 2 + l_2^2 (k_0^2 \mu_2 \epsilon_2 - k_i^2) + 4 l_1^2 (k_0^2 - k_i^2)
\]

\[
+ \left( \epsilon_2 (k_0^2 - k_i^2) + \frac{1}{\epsilon_2} (k_0^2 \mu_2 \epsilon_2 - k_i^2) \right) (2 l_1 l_2)
\]

(20)

Similar expansion can also be carried out to first order.

C. Thin Layers

Let us consider a single dielectric layer with \(\epsilon_i = 1\) and \(l_i \rightarrow 0\) or \(k_i \rightarrow 0\). As is well-known, first-order series expansion shows that this corresponds to approximating the transmission line by shunt capacitor \(C_i = l_i \epsilon_i\). In the field picture, this approximation represents a dipole approximation for the dielectric layer. This approximation is exact for all frequencies in the limit \(l_i \rightarrow 0\), \(\epsilon_i l_i \rightarrow C_i\). Similarly, a thin magnetic layer, or layer at very low frequency corresponds to a series inductance \(L_i = l_i \mu_i\).

D. Homogenization/Effective Medium Approximation

Homogenization approximation is low-frequency approximation used to obtain effective material parameters. Generally, this involves averaging the fields over the unit cell [12], [13]. An equivalent formulation can be obtained by series expansion of the dispersion relation [1]. It comprises the application of the quasi-static approximation to the host material and to the Bloch waves \(\cos k_B l \approx 1 - \frac{k_B^2 l^2}{2}\). In this case the 1-D MTM behave as effective uniaxial materials [1].

Effective constants can be “derived” by lumped element approximations to the transmission line representation described in Sec. IV-C. For example, for the single layer material, the effective transverse permittivity \(\epsilon_i\) can be approximated a single capacitor \(C_i = \epsilon_i l\). It is equivalent to the three capacitors in parallel so that:

\[
\epsilon_i l = \epsilon_i l_1 + \epsilon_2 l_2 + \epsilon_1 l_1.
\]

(21)

The permittivity in the direction along the axis of stratification \(\epsilon_n\) can be deduced by treating the capacitors in series since the electric field is now perpendicular to the layer.

\[
\frac{1}{\epsilon_n} = \frac{l_1}{\epsilon_1} + \frac{l_2}{\epsilon_2} + \frac{l_1}{\epsilon_1}.
\]

(22)

The advantage of the lumped circuit model is that it is easily extensible to multiple layers. A similar method works for
the effective permeability, except that we must use duality substitutions, so that
\[ \mu_{il} = \mu_{l1} + \mu_{l2} + \mu_{l1}, \] (23a)
\[ \frac{1}{\mu_n} = \frac{l_1}{\mu_1} + \frac{l_2}{\mu_2} + \frac{l_1}{\mu_1}, \] (23b)

The dispersion relation and the characteristic impedance for
the uniaxial media are [1]:
\[ k_B = \begin{cases} \frac{\mu_t}{\varepsilon_n} k_0^2 \varepsilon_n \mu_t - k_0^2 & \text{TM mode,} \\ \frac{\mu_t}{\varepsilon_n} (k_0^2 \varepsilon_n \mu_t - k_0^2) & \text{TE mode,} \end{cases} \] (24a)
\[ Z_B = \begin{cases} \frac{\mu_t}{\varepsilon_n} \sqrt{1 - \frac{k_t^2}{k_0^2 \varepsilon_n \mu_t}} & \text{TM mode,} \\ \frac{\mu_t}{\varepsilon_n} \sqrt{1 - \frac{\mu_t^2}{\mu_n^2 \varepsilon_n \mu_t}} & \text{TE mode,} \end{cases} \] (24b)

E. Extended Effective Medium Approximation

Our goal is to obtain approximations that apply over frequency range greater than that of the homogenization approximation. In particular, it is possible that the QSA may hold for each individual layers but not for the Bloch waves. We can use this to obtain a good approximation valid from dc to the band edge. We use the QSA for the layers but we do not expand the term \( \cos k_B l \). Then we observe that the right side of such equation forms a second-order polynomial in \( k_t \) and \( k_0 \). This polynomial is the result of homogenization of the fields and corresponds to the well-known dispersion relations for uniaxial crystals [1]. Thus, we can obtain a general approximation as
\[ k_B = \begin{cases} \frac{1}{\mu_t} \cos^{-1} \left[ 1 + \frac{\mu_t^2}{\mu_n^2} \varepsilon_n \mu_t - k_0^2 \varepsilon_n \mu_t \right] & \text{TM mode,} \\ \frac{1}{\mu_n} \cos^{-1} \left[ 1 + \frac{\mu_t^2}{\mu_n^2} (k_0^2 \varepsilon_n \mu_t - k_0^2) \varepsilon_n \mu_t \right] & \text{TE mode,} \end{cases} \] (25)

where, the parameters \( \varepsilon_t \) and \( \varepsilon_n \) can be derived conveniently using the lumped element equivalent circuits as described in Sec. IV-C. We can use this value of \( k_B \) in the impedance formula Eq. 24b together with a factor that gives the correct frequency dependence at the band boundary \( (\sqrt{1 - k_B^2}/\pi^2 \) for TM modes with \( \varepsilon_2 > 1 \) so that the impedance can be written as follows:
\[ Z_B = \begin{cases} \frac{k_B}{\omega_\ell} f(k_B^2) & \text{TM mode,} \\ \frac{k_B}{\omega_\ell} f(k_B^2) & \text{TE mode,} \end{cases} \] (26)

F. Approximation Examples

We performed a large number of simulations to verify the accuracy of the approximations. Figures 5 through 9 show the comparison of exact and approximate results plotting the impedance and the dispersion relations. The figures show real and imaginary parts of both quantities. The notation on the legend for the approximate results is the following:

- A plain variable represents an exact expression,
- A variable with a "hat" (\( ^\wedge \)) represents the extended effective medium approximation, and
- A variable with a "tilde" (\( ^\sim \)) represents the effective medium approximation.
This case, the plain effective medium approximation is accurate as the effect of the inclusion is not felt except at the zone boundary. Thus, we can use either the effective medium or the extended effective medium approximation with reasonable accuracy.

V. Application to Surface Waves

The problem of determining the dispersion relation for surface waves is important for antenna and other applications. There have been several investigations analyzing properties of surface waves on a MTM slab [14], [15]. These analysis treated the MTM as a homogeneous material. In this section we apply the approximate solutions to modal decomposition to obtain the dispersion relation.

A. Homogeneous slab

First, we review the solution for a homogeneous slab. We consider a homogeneous slab with permittivity $\varepsilon_s$, permeability $\mu_s$, and thickness $d$. The slab interfaces with a perfect conductor on one side and air on the other side. The direction of propagation of the surface wave is along the interface axis ($x$ axis), but we solve the problem using transverse resonance taking the direction of propagation for the resonance problem transverse to the interface. Fig. 10 shows the configuration and the transverse resonance circuit.

We perform our calculation for TM modes. We find dispersion relation by transverse resonance, $Z_a = Z_s$, which corresponds to a situation where the sum of the impedances looking to the left and right of the interface is equal to zero.

$$Z_0 + jZ_s \tan \kappa_s d = 0. \quad (27)$$

We substitute Eq. 4 into Eq. 27 to obtain:

$$\frac{|k_a|}{\omega} = \frac{k_s}{\omega \varepsilon_s} \tan k_s d \quad (28a)$$

$$|k_a| = \sqrt{k_t^2 - k_0^2} \quad (28b)$$

$$k_s = \sqrt{k_t^2 - k_0 \varepsilon_s} \quad (28c)$$
The surface wave modes occur only if $k_a$ is imaginary so that $k_a = j|k_a|$. We wish to find dispersion relation for the guide wavelength

$$\frac{\lambda_0}{\lambda_g} = \frac{k_t}{k_0}$$  \hspace{1cm} (29)

We introduce the following dimensionless quantities:

$$q = |k_a|d, \quad p = k_s d, \quad v = k_t d, \quad u = k_0 d.$$  \hspace{1cm} (30)

Thus, we need to solve the following three equations in three unknowns $p$, $q$, and $v$:

$$q = \frac{p}{\epsilon_s} \tan p,$$  \hspace{1cm} (31a)

$$q = \sqrt{u^2 - v^2},$$  \hspace{1cm} (31b)

$$p = \sqrt{v^2 \epsilon_s - u^2}.$$  \hspace{1cm} (31c)

We can obtain the dispersion relation from Eq. 31 directly since we can compute $q$ for any value of $p$ and then solve for $u$ and $v$.

**B. Metamaterial Slab**

Now we attempt to find the dispersion relation for the MTM slab. Again, we consider MTM slab with thickness $d$ over ground plane and interface with air. Using transverse resonance again, we need to solve:

$$\frac{j k_a}{\omega} = Z_B \tan k_B d$$  \hspace{1cm} (32a)

$$|k_a| = \sqrt{k_t^2 - k_0^2}$$  \hspace{1cm} (32b)

$$k_B = \mathcal{F}(k_0, k_t) = \mathcal{F}(k_a, k_t)$$  \hspace{1cm} (32c)

where $Z_B$ and $k_B$ have some complex dependence on the external parameters.

If we attempt to use the full-wave solution, the problem is not soluble analytically. We can use numeric methods. The QSA for all or any of the layers does not help because it does not allow us to eliminate the internal material variables $k_i$. However, if we use the effective medium or the extended effective medium approximation we can solve the problem analytically. In particular, the extended effective medium approximation replicates the relevant features of the band structure yet yields an analytic solution. Fig. 11 shows some typical results.

**VI. CONCLUSIONS**

We determined the exact modal structure of a one-dimensional electromagnetic crystal and constructed an equivalent transmission line model. Then, we derived approximations that replicated the band structure of this metamaterial, but is expressed purely in terms of external parameters, such as frequency, transverse wavenumber and the Bloch wavenumber. In addition to the well-known effective medium approximation, we showed an extended effective medium approximation that allows us to compute the impedance and the dispersion relation over frequency regions where the effective medium approximation is not valid. Thus, we are able to determine useful engineering quantities, while avoiding the need to determine effective parameters.

We believe this approach can be extended to other types of MTMs, including double-negative materials in order to elucidate their behavior over wider frequency ranges.

**REFERENCES**


