Scalar Potential Depolarizing Dyad Artifact for a Uniaxial Bianisotropic Medium

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Abstract – A scalar potential formulation for a homogeneous uniaxial bianisotropic medium is derived through the use of Helmholtz’s theorem and operator orthogonality and compared with prior findings. It is shown that an expected and unexpected depolarizing dyad appears in the development. Using a spectral domain analysis and Leibnitz’s rule in the field recovery process, it is shown the unexpected depolarizing dyad is canceled. Thus, a mathematically and physically consistent scalar potential theory is confirmed.

I. INTRODUCTION

Vector and scalar potentials are often used in analyzing electromagnetic problems involving simple [1] and complex [2] media such as uniaxially anisotropic [3] and bianisotropic [4] materials. These media are interesting from an application viewpoint, thus a detailed understanding of scalar potential formulations involving complex media is warranted. The goal here is to develop a scalar potential formulation for a uniaxial bianisotropic medium based on Helmholtz’s theorem and operator orthogonality. Comparison with prior findings reveals subtle differences that have important ramifications in field uniqueness. In addition, in this development and in prior findings, both physically expected and unexpected depolarizing dyads are identified. It is shown here that the unexpected depolarizing dyad is canceled via a spectral domain analysis and careful application of Leibnitz’s rule in the field recovery process, resulting in a mathematically and physically consistent theory.

II. SCALAR POTENTIAL FORMULATION

Maxwell’s curl equations for a linear, homogeneous, bianisotropic uniaxial medium are \( \epsilon \omega \) assumed

\[
\nabla \times \vec{E}(\rho, z) = -j \omega \mu_0 \vec{H}(\rho, z) - j \omega \epsilon_0 \vec{E}(\rho, z) \\
\nabla \times \vec{H}(\rho, z) = \vec{j}_e(\rho, z) + j \omega \epsilon_0 \vec{E}(\rho, z) + j \omega \mu_0 \vec{H}(\rho, z)
\]

where \( \sigma = \sigma_\parallel + \frac{\sigma_\perp}{2} \) for \( \sigma = \epsilon, \mu, \xi, \zeta \) and \( \vec{l}_z = \hat{\mathbf{x}} + j \hat{\mathbf{y}} \). Decomposing (1) into transverse and longitudinal components leads to the following Faraday and Ampere law relations (with \( \nabla_t \) the transverse gradient operator)

\[
\nabla_t \times \vec{E}_z + \frac{\partial \vec{E}_z}{\partial z} = -\vec{j}_h - j \omega \mu_0 \vec{H}_t - j \omega \epsilon_0 \vec{E}_t, \quad \nabla_t \times \vec{H}_z = \frac{\partial \vec{H}_z}{\partial z} - j \omega \epsilon_0 \vec{E}_z + j \omega \mu_0 \vec{H}_z \\
\nabla_t \times \vec{H}_z + \frac{\partial \vec{H}_z}{\partial z} = \vec{j}_e - j \omega \epsilon_0 \vec{E}_t + j \omega \mu_0 \vec{H}_t, \quad \nabla_t \times \vec{H}_z = j \omega \epsilon_0 \vec{E}_z + j \omega \mu_0 \vec{H}_z
\]

Based on the 2D form of Helmholtz’s theorem, the transverse field and current densities can be represented as

\[
\vec{E}_t = \nabla_t \Phi + \nabla_t \times \vec{z}\theta = \nabla_t \Phi - \vec{z} \times \nabla_t \theta, \quad \vec{H}_t = \nabla_t \pi + \nabla_t \times \vec{z}\psi = \nabla_t \pi - \vec{z} \times \nabla_t \psi \\
\vec{j}_e = \nabla_t \mu_e + \nabla_t \times \vec{z}\nu_e = \nabla_t \mu_e - \vec{z} \times \nabla_t \nu_e, \quad \vec{j}_h = \nabla_t \mu_h + \nabla_t \times \vec{z}\nu_h = \nabla_t \mu_h - \vec{z} \times \nabla_t \nu_h
\]

where \( \Phi, \pi, \psi, \mu_e, \mu_h, \nu_e, \nu_h \) are scalar potentials. Inserting (4) into the second relations of (2) and (3) gives

\[
\begin{bmatrix}
\vec{E}_z \\
\vec{H}_z
\end{bmatrix} = 
\begin{bmatrix}
-\frac{\mu_e}{j \omega \epsilon_0} \nabla_t^2 \psi - \frac{\epsilon_e}{j \omega \mu_0} \nabla_t^2 \theta - \epsilon_e \vec{l}_z \cdot \vec{j}_e + \epsilon_e \vec{l}_h \cdot \vec{j}_h \\
-\frac{\varepsilon_e}{j \omega \mu_0} \nabla_t^2 \psi + \frac{\mu_e}{j \omega \epsilon_0} \nabla_t^2 \theta + \mu_e \vec{l}_z \cdot \vec{j}_e + \mu_e \vec{l}_h \cdot \vec{j}_h
\end{bmatrix}
\]

\[
\vec{l}_z = \vec{z}, \quad \nabla_t \cdot \vec{l}_z = 0, \quad \vec{l}_h = \vec{z}, \quad \nabla_t \cdot \vec{l}_h = 0
\]
where \( \tilde{F}^{ee}, \tilde{F}^{eh}, \tilde{F}^{he}, \tilde{F}^{hh} \) are the well-known (i.e., physically/mathematically expected) longitudinal depolarizing dyads that occur via careful handling of the source singularity [5]. That is, an excavated slice gap region along \( z \) is placed around the source to avoid the singularity, but in so doing the longitudinal currents are disrupted and surface are charges subsequently induced. Consequently, gap fields are produced which are not there in the original continuous volume current region and thus the longitudinal depolarizing dyads in (6) are implicated to nullify these gap fields. Note, the field recovery process is computed from the potentials via (4) and (6).

The governing relations for \( \Phi, \theta, \pi, \psi \) may be found via direct substitution of (4)-(5) into the first relations of (2)-(3). This contrasts prior work in which the transverse divergence and curl of (2)-(3) are performed (see [3], for example). Noting that \( \nabla \times \nabla \Phi = \nabla \Phi \) and \( \nabla \times \nabla \theta = \nabla \theta \), it follows that the terms in the first relations of (11) are canceled, which is the method preferred here. Fourier transformation of (13) from the \( \Omega -\)domain (denoted by a tilde) via

\[
\nabla \times \nabla \Phi = \nabla \Phi \]

\[
\nabla \times \nabla \theta = \nabla \theta \]

where \( \tilde{E}_i = \nabla \Phi, \tilde{H}_i = \nabla \pi \) and \( \tilde{J}_{el} = \nabla \mu_e, \tilde{J}_{hl} = \nabla \mu_h \) are the transverse laminar (i.e., curl-free) fields and their densities and the transverse depolarizing dyads are \( \tilde{F}^{ee}, \tilde{F}^{eh}, \tilde{F}^{he}, \tilde{F}^{hh} \). These dyads, in contrast to the longitudinal depolarizing dyads in (6), are unexpected since currents parallel to the gap don’t induce charges, thus no gap fields are implicated. It is this physical picture that prompted the author to seek a mathematical analysis that demonstrates cancelation of these dyad terms, as will be shown in the next section. Finalizing, substitution of (6),(11) into (8) and (10) leads to the following coupled differential equations for \( \psi \) and \( \theta \)

\[
\left[ \begin{array}{c} L_1 \\ L_2 \\ L_3 \\ L_4 \end{array} \right] \left[ \begin{array}{c} \psi \\ \theta \end{array} \right] = \left[ \begin{array}{c} s_1 \\ s_2 \end{array} \right]
\]

III. DEMONSTRATION OF TRANSVERSE DEPOLARIZING DYAD CANCELATION

A spectral domain analysis is used to mathematically show that the transverse depolarizing dyads in the second relations of (11) are canceled. Equivalently, one can show the \( u_e \) and \( u_h \) terms in the first relations of (11) are canceled, which is the method preferred here. Fourier transformation of (13) from the \( \rho, \sigma \) to the \( \tilde{\rho}, \tilde{\sigma} \) -domain and subsequent inverse transformation back to the \( \tilde{\rho}, \tilde{\sigma} \) -domain (denoted by a tilde) via
complex plane analysis leads to the following expression for $\bar{\psi}(\bar{\lambda}_\rho, z)$ (assuming currents exist in the region $a < z < b$)

$$\bar{\psi} = \int_a^b \left[ \frac{e^{j\mu_0 \lambda_\mu^2 \rho z}}{h_\mu \lambda_\mu} \int_j \frac{e^{-j\mu_0 \lambda_\mu^2 \rho z}}{h_\mu \lambda_\mu} \right] e^{-j/\lambda_\rho |z-z'|} dz'$$

where $\pm\lambda_{a\rho}$, $\pm\lambda_{b\rho}$ are the roots of $L_2 L_4 - L_2 L_5 = 0$ in the transform domain and $\bar{\lambda}_\rho = \hat{\lambda}_a + \hat{\lambda}_b$. Note, a similar expression for $\hat{\theta}$ holds via duality and is omitted here for the sake of brevity. Computing $\Phi$ via Fourier transformation of (11), it is noted that the derivative of $\bar{\psi}(\bar{\lambda}_\rho, z)$ with respect to $z$ is required. Care must be taken in performing this derivative since the integral must be broken into two subintervals $[a, z)$ and $(z, b]$. That is, Leibnitz’s rule must be used to compute the derivatives, namely

$$\frac{d}{dz} \int_a^b f(z, z')dz' = \int_a^b \frac{d}{dz} f(z, z')dz' + \int_a^b f(z, z') \frac{d}{dz} z' dz'$$

Remarkably, upon computing the derivative of $\bar{\psi}$ using (15), an additional $-u_c$ term is produced that exactly cancels the unexpected $u_c$ term in (11)! Duality can be used to show the unexpected term $+u_h$ is removed.

VI. CONCLUSION

A scalar potential formulation, based on Helmholtz’s theorem and operator orthogonality, was presented for a uniaxial bianisotropic medium. Field uniqueness via comparison to prior findings was discussed. Unexpected depolarizing dyads present in this formulation (and found in prior findings) were shown to be canceled via careful application of Leibnitz’s rule. Thus, a mathematically and physically consistent theory was confirmed.

ACKNOWLEDGEMENT

The views expressed in this article are those of the author and do not reflect the official policy or position of the United States Air Force, Department of Defense, or the U.S. Government.

REFERENCES