Sum of Squares Based Nonlinear Control Design for Diesel Engine *

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Abstract—In this paper we propose a nonlinear controller based on sum of squares decomposition applied to turbocharged diesel engine. The basic idea is to employ polynomial Lyapunov function in order to formulate the stability sufficient condition in the terms of state dependent matrix inequalities. The obtained controller gain guarantees the global convergence of the system and regulates the flows for the Variable Geometry Turbocharger and the Exhaust Gas Recirculation systems in order to minimize the NOX emission and the smoke of diesel engine. Simulation of the control performances through a professional software shows the effectiveness of this approach.

I. INTRODUCTION

This paper develops nonlinear control design for a diesel engine equipped by Exhaust Gas Recirculation (EGR) and Variable Geometry Turbine (VGT) systems. The objective of the controller is to manage EGR-VGT actuators for a given operating conditions that allows to follow low fuel consumption and toxic exhaust emissions according to ecological requirements [1]. There are main reasons that diesel engines are solutions for technic applications due to their low fuel consumption and durability. Therefore, this is important to develop optimal control strategies of automotive diesel engine by using model based modern technologies [2], [3], [4], [5]. In order do design the nonlinear control of diesel engine, various approaches have been done. References [7] and [8] present an overview of different control aspects of Diesel engines with EGR. Known technique has been applied in [9] about fuzzy identification of nonlinear control systems and a linear matrix inequality approach that is widely used for nonlinear control problems. For example, in the paper [10], the control strategy of diesel engine air path described by Takagi-Sugeno (TS) model is presented. Similar work has been done by [11] in order to design fuzzy robust tracking control with pole placement. Gain scheduling control is used often, see for example [12]. For nonlinear dynamical systems, Lyapunov function based methods play an important role in both stability analysis and control synthesis. The diesel engine control synthesis typically involves design of the control Lyapunov function. Our paper extends the work of [12], which presents a Control Lyapunov Function approach. This approach has been used also in [13][14]. But in contrary to [12] based on quadratic Lyapunov functions, we extend the form of Lyapunov functions to rational matrix functions of states.

In this paper we propose a new way to apply Sum-of-Squares (SOS) technic approach that has been used in order to design the nonlinear polynomial control for diesel engine. The control design procedure leads to SOS problem which can be solved efficiently with Sum-of-Squares programming and SOSTOOLS (see [15]). SOS is a powerful and promising technique which has been widely used in recent years by [16], [17], [18] and others. However, its potential for nonlinear control system synthesis has not been fully explored. Therefore, the usage of newly emerging SOS decomposition for design of optimal diesel engine control is an unique feature. The proposed approach extends the quadratic Lyapunov function case considered in [12] to more general polynomial Lyapunov function that helps to search for a best fit Lyapunov function and construct the resulting nonlinear control law so that the design objective can be optimized. We discuss the advantages of nonlinear SOS-based controller in order to evaluate the experimental benefits using an advanced diesel engine professional simulator AMESim (LMS).

This paper is organized as follows. In section 2 the turbocharged diesel engine has been described and the dynamical model is introduced. After having pointed the control design problem and the methodology in section 3, we apply of SOS condition in section 4. A construction of control Lyapunov function is presented and a control law has been designed in section 5. This control strategy is successfully evaluated using AMESim virtual testbed in section 6. Conclusion and recommendations for future work are presented in the last section.

II. DESCRIPTION OF THE MODEL

In this section, we present a model of a diesel engine equipped with VGT and EGR valves. We consider a three order model for the intake flow loop based on the simplifying assumptions that all thermodynamic properties are referenced to air and the intake and exhaust manifold temperature dynamics are neglected. The modeling effort is focused on the gas flows. In this paper we follow [4] to model a cylinder flow and a mass flow through the EGR and VGT systems. More details of modeling can be seen in [7] at which this approach has been adopted previously.

The model proposed here has three states: the intake manifold pressure $p_{im}$, the exhaust manifold pressure $p_{em}$, and the compressor power $P_c$ which describe the main dynamics and the most important system properties. The turbocharger dynamic is modeled as a first-order lag power transfer with time constant $\tau$. These states are collected in a
TABLE I
NOMENCLATURE

<table>
<thead>
<tr>
<th>Symbol</th>
<th>Description</th>
</tr>
</thead>
<tbody>
<tr>
<td>AF</td>
<td>Air-fuel ratio</td>
</tr>
<tr>
<td>EGR</td>
<td>EGR flow fraction defined as ( \frac{W_{egr}}{W_{egr} + W_{c}} )</td>
</tr>
<tr>
<td>( \eta_{vol} )</td>
<td>Volumetric efficiency</td>
</tr>
<tr>
<td>( N )</td>
<td>Engine speed</td>
</tr>
<tr>
<td>( n_{cyl} )</td>
<td>Number of cylinders</td>
</tr>
<tr>
<td>( V_f )</td>
<td>Volume</td>
</tr>
<tr>
<td>( R_a )</td>
<td>Gas constant of the air</td>
</tr>
<tr>
<td>( W_c )</td>
<td>Compressor mass flow</td>
</tr>
<tr>
<td>( P_{amb} )</td>
<td>Ambient pressure</td>
</tr>
<tr>
<td>( T_{amb} )</td>
<td>Ambient temperature</td>
</tr>
<tr>
<td>( I_f )</td>
<td>Turbocharger moment of inertia</td>
</tr>
<tr>
<td>( W_f )</td>
<td>Engine fueling rate requested by the driver</td>
</tr>
<tr>
<td>( \tau )</td>
<td>Time constant</td>
</tr>
<tr>
<td>( V_{im} )</td>
<td>Intake manifold volume</td>
</tr>
<tr>
<td>( V_{em} )</td>
<td>Exhaust manifold volume</td>
</tr>
<tr>
<td>( p_{im} )</td>
<td>Intake manifold pressure</td>
</tr>
<tr>
<td>( p_{em} )</td>
<td>Exhaust manifold pressure</td>
</tr>
<tr>
<td>( T_{im} )</td>
<td>Intake manifold temperature</td>
</tr>
<tr>
<td>( T_{em} )</td>
<td>Exhaust manifold temperature</td>
</tr>
<tr>
<td>( T_e )</td>
<td>Temperature after the compressor</td>
</tr>
<tr>
<td>( A_{opt \text{ max}} )</td>
<td>Maximum area in the turbine that gas flow through</td>
</tr>
<tr>
<td>( A_{egr \text{ max}} )</td>
<td>Maximum effective area</td>
</tr>
<tr>
<td>( \gamma_e )</td>
<td>Specific heat capacity ratio of exhaust gas</td>
</tr>
<tr>
<td>( \gamma_a )</td>
<td>Specific heat capacity ratio for air</td>
</tr>
<tr>
<td>( R_e )</td>
<td>Exhaust gas constant</td>
</tr>
<tr>
<td>( c_{pe} )</td>
<td>Specific heat capacity at constant pressure for exhaust gas</td>
</tr>
<tr>
<td>( c_{pa} )</td>
<td>Specific heat capacity at constant pressure for air</td>
</tr>
<tr>
<td>( \eta_t )</td>
<td>Turbine isentropic efficiency</td>
</tr>
<tr>
<td>( \eta_{mech} )</td>
<td>Turbocharger mechanical efficiency</td>
</tr>
<tr>
<td>( \eta_k )</td>
<td>Compressor isentropic efficiency</td>
</tr>
<tr>
<td>( \eta_t )</td>
<td>Turbocharger efficiency</td>
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<tr>
<td>( \eta_{cyl} )</td>
<td>Compressor efficiency</td>
</tr>
<tr>
<td>( W_{im} )</td>
<td>Intake manifold pressure</td>
</tr>
<tr>
<td>( W_{em} )</td>
<td>Exhaust manifold pressure</td>
</tr>
<tr>
<td>( W_c )</td>
<td>Compressor air mass flow rate</td>
</tr>
<tr>
<td>( W_t )</td>
<td>Turbine gas mass flow rate</td>
</tr>
<tr>
<td>( W_{egr} )</td>
<td>EGR mass flow rate</td>
</tr>
<tr>
<td>( W_f )</td>
<td>Engine fueling rate requested by the driver</td>
</tr>
</tbody>
</table>

We assign \( v_1 = W_{egr} \) and \( v_2 = W_{t} \) which are the two control inputs with the assumption that the desired flow values can be assigned by manipulating the EGR and the VGT actuators \( k_{egr} \) and \( k_{vgt} \). In this approach, it is necessary to invert the flow rates to obtain the EGR valve position and the VGT vane position commands that are the control inputs to the model.

A. Feedback Transformation and Outputs

The first step in our control method is to make feedback linearization which is done by [7], [14]. We employ input-output linearization for the diesel engine model. Thus, consider the system (1) with the output vector defined as

\[
y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix} = \begin{bmatrix} W_c - W_{\text{egr}}^d \\ p_{em} - p_{\text{egr}}^d \end{bmatrix}. \tag{3}
\]

It was shown in [7] that the system has an equilibrium which is stable. With the chosen outputs we obtain the following equations

\[
\begin{align*}
y_1 &= -a \left( y_1 + W_{c}^d - k_e p_{im} \right) - \frac{1}{2} \left( y_1 + W_{c}^d \right) - a v_1 + b v_2 \\
y_2 &= k_e \left( k_e p_{im} + W_f \right) - k_e v_1 - k_e v_2 \\
p_{im} &= k_1 \left( y_1 + W_{c}^d - k_e p_{im} + v_1 \right)
\end{align*}
\]

where

\[
a = k_1 \mu \left( p_{im} \right)^{\mu-1} \left( y_1 + W_{c}^d \right) / \left( \left( p_{im} \right)^{\mu} - 1 \right), \quad b = \frac{1}{\tau} \eta^* \frac{T_a}{T} \left( \frac{p_{em}}{p_{im}} \right)^{\mu} - 1, \tag{5}
\]

and \( \eta^* = \eta_m \eta_t \eta_{mech} \).

To establish the stability of the internal dynamics, we examine the zero dynamics of the system as follows \( y = \dot{y} = 0 \) and solve for \( v_1, v_2 \). Then after substitute this into \( p_{im} \) dynamic, we obtain stable zero dynamic for the equilibrium \( p_{im}^e \). With the zero dynamic stable, the standard stabilization technique for input-output linearizable system can be applied. The feedback transformation \( v = G^{-1} \left( w - F \right) \) with invertible matrix

\[
G = \begin{bmatrix} -a & b \\ -k_2 & -k_2 \end{bmatrix}
\]

yields

\[
\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = G^{-1} \left( w_1 + a \left( y_1 + W_{c}^d - k_e p_{im} \right) + \frac{1}{2} \left( y_1 + W_{c}^d \right) \right) \frac{w_2 - k_e p_{im} - W_f}{w_2 - k_e p_{im} - W_f}. \tag{6}
\]

Introducing a new coordinate \( z = p_{im}^{\mu} - \left( p_{im}^e \right)^{\mu} \) and renders from new inputs \( w_1, w_2 \) to the outputs \( y_1, y_2 \) the system becomes

\[
\begin{align*}
\dot{y}_1 &= w_1 \\
\dot{y}_2 &= w_2 \\
\dot{z} &= k_e \left( b - \frac{1}{\tau} \right) y_1 + f_z - k_e \dot{w}_1 - k_e \dot{w}_2
\end{align*}
\]

where

\[
k_e = \mu \left( p_{im} \right)^{\mu-1} \frac{k_1}{b + a}, \tag{7}
\]

\[
f_z = k_e \left( b - \frac{1}{\tau} \right) W_{c}^d + b k_e W_f + (k_2 - k_e) p_{im} b. \tag{8}
\]

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III. CONTROL METHODOLOGY

A. Control Objectives

The design objectives are to regulate the air-fuel ratio and the EGR fraction to their respective setpoints determined from the engine data $A_{\text{ref}} = A_{\text{ref}}(N, W_f)$, $EGR_{\text{ref}} = EGR_{\text{ref}}(N, W_f)$. The generated static maps based on a trade-off between maximum fuel economy and minimal $NO_x$ and smoke generation. The setpoints of air-fuel ratio and the EGR fraction can be transformed into the setpoints of the compressor and the EGR mass flow rates $W_C^d$ and $W_{egr}^d$ using their relationship in steady state. See [7] for details.

B. System Stabilization and Control Synthesis

Consider a polynomial nonlinear system of the following form

$$\dot{x} = A(x)x + B(x)u$$

(10)

where the system state $x \in \mathbb{R}^n$, control input $u \in \mathbb{R}^m$. It is assumed that the state-space matrices $A(\cdot), B(\cdot)$ are polynomial functions of the state $x$ and $(x, u) = (0, 0)$ is an equilibrium of the nonlinear system (10). We design a static nonlinear state feedback controller $u = F(x)$ such that the nonlinear system (10) is stabilized asymptotically in a finite state region. In general, $F(x)$ will be a rational function of $x$. The control synthesis involves design of the control Lyapunov function and the nonlinear controllers.

To apply this control law for the system (4) we center the controls $v_1$ and $v_2$ at their setpoint values and obtain

$$v_1 = u_1 + W_{egr}^d, \quad v_2 = u_2 + (W_C^d + W_f).$$

(11)

Previously, nonlinear control designs usually are based on quadratic Lyapunov functions. In this paper, we relax the form of Lyapunov functions to rational matrix functions of states in the form of $V(x) = x^T P(x)x$, $P(x) > 0$ Lyapunov level set can be used to estimate the domain of attraction for regional stabilization. Note that the Lyapunov matrix $P(x)$ is state-dependent. For the given state feedback control law, the closed-loop system becomes

$$\dot{x} = [A(x) + B(x)F(x)]x.$$

(12)

We specify the state derivative bound as $m = [m_1, m_2, \ldots, m_n]^T$ and enforce the state derivative constraints $|\dot{x}_i| = |e_f^T [A(x) + B(x)F(x)]| \leq m_i, \quad i \in \{1, \ldots, n\}$, where $e_i$ is the $i$-th column of the identity matrix $I_n$.

By summarizing the above discussions, we have the following theorem for synthesis of a nonlinear state feedback control law using rational Lyapunov functions.

**Theorem 1.** Given the state derivative bounds $m$ and a small positive number $\varepsilon$, the nonlinear system (10) is asymptotically stabilizable by a nonlinear state feedback controller $u = F(x)x$ if there exist polynomial matrix functions $Q(x) \in \mathbb{S}_{+}^{m \times n}$, $K(x) \in \mathbb{R}^{m \times n}$ such that

$$Q(x)A^T(x) + A(x)Q(x) + B(x)K(x) + K^T(x)B^T(x) \geq \sum_{i=1}^{n} \left( m_i \frac{\partial Q(x)}{\partial x_i} \right) < 0$$

(14)

Furthermore, the stabilizing rational Lyapunov function is $V(x) = x^T Q^{-1}(x)x$ and its associated nonlinear controller gain given by $F(x) = K(x)Q^{-1}(x)$.

**Proof:** The proof methodology is based on the work [17]. Since $Q^{-1}(x) > \varepsilon I$, the set $\Omega(Q^{-1}, 1)$ will specify a finite state region containing origin. Function $V(x) = x^T Q^{-1}(x)x > \varepsilon x^T x$ is positive definite. For the Lyapunov function $V(x) = x^T Q^{-1}(x)x$ and the control law $u = F(x)x$, we have the local stabilization conditions for the system (10)

$$V(x) > 0$$

(15)

$$V(x) < 0$$

(16)

$$\{x^T Q^{-1}(x)x \leq 1 \} \subset \bigcap_{i=1}^{n} \left| e_i^T [A(x) + B(x)F(x)]x \right| \leq m_i$$

(17)

From equation (14), we get

$$QA^T + AQ + BK + K^T B^T = \sum_{i=1}^{n} \left( \dot{x}_i \frac{\partial Q(x)}{\partial x_i} \right) < 0$$

(18)

for any $|\dot{x}_i| \leq m_i$. Multiplying $Q^{-1}(x)$ from left and right hand sides, confirm that $V < 0$ where $K(x) = F(x)Q(x)$.

Note that conditions (13)-(14) are state-dependent polynomial matrix inequalities. Using SOS decomposition and semi-definite programming, they can be solved by parameterizing $Q(x)$ and $K(x)$ in proper polynomial forms. Nevertheless, choosing different Lyapunov function forms could result in different shapes and sizes of Lyapunov level set. To render (13)-(14) into computationally tractable SOS conditions, it often necessary to add some regional constraints in the form of $R_j(x) < 0, \quad j \in I[1, r]$. Then the modified state feedback stabilization condition in terms of SOS will be

$$-Z_1^T \left( Q(x) - \frac{1}{\varepsilon} I \right) Z_1 + \sum_{j=1}^{r} \lambda_{1,j}(z_1, x) R_j(x) \in \Phi_{\text{SOS}}$$

(19)

$$-Z_2^T \left[ Q(x)A^T(x) + A(x)Q(x) + B(x)K(x) + K^T(x)B^T(x) + \sum_{i=1}^{n} \left( \dot{x}_i \frac{\partial Q(x)}{\partial x_i} \right) \right] Z_2 + \sum_{j=1}^{r} \lambda_{2,j}(z_2, x) R_j(x) \in \Phi_{\text{SOS}}$$

(20)

where $Z_1, Z_2$ are free variables. $\lambda_{1,j}, \lambda_{2,j}, \lambda_{3,j} > 0$ are SOS multipliers. Clearly, this will provide a sufficient condition for equations (13)-(14) and its solution will be valid in a local region specified by the intersection of constraints $R_j(x) < 0, \quad j \in I[1, r]$. Note that $R_j(x) < 0$ could be any user-specified function of $x$ which characterizes a closed region of states.
IV. APPLICATION OF SOS CONDITION

Consider the system (7) and rewrite it into the matrix form (10) and find the control law of the form \( u = F(x)x \).

The system becomes

\[
\dot{x} = (A(x) + B(x)F(x))x
\]

(21)

where \( x = (y_1, y_2, z)^T \), \( u = (w_1, w_2)^T \) and the matrices have state-depending form

\[
A(x) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
a_{31}(x) & a_{32}(x) & 0
\end{pmatrix},
B(x) = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
b_{31}(x) & b_{32}(x)
\end{pmatrix}
\]

where \( a_{31}(x) = k_e \left( b - \frac{1}{\tau_a} \right) \), \( a_{32}(x) = \frac{f_i(y_1)}{y_2} \), \( b_{31}(x) = -k_e \), \( b_{32}(x) = -\frac{k_b}{\tau_m} \). And \( f_{ij} = f_{ij}(x) \), \( i = 1, n \) are components of the matrix \( F(x) \).

Using definition (8), (9) of the functions \( k_e \) and \( f_e \), the matrices components can be rewritten in the form:

\[
a_{31}(x) = \frac{k_1}{\eta \tau_a} \mu^{r_{im} - 1} \left( 1 - \eta \left( \frac{T_m}{\tau_a} \right)^{r_{im}} \right),
\]

\[
a_{32}(x) = \frac{f_i(y_1)}{y_2} = \frac{k_1}{\eta} \left( 1 - \frac{T_m}{\tau_a} \right) W_d + \frac{k_2}{\tau_a} W_f + (k_e - k_2)p_{im},
\]

\[
b_{31}(x) = -\mu^{i_{im}} k_1, \quad b_{32}(x) = \frac{\mu^{i_{im}}}{\eta \tau_a} \frac{k_1}{\eta^{r_{im}}} \left( 1 - \frac{T_m}{\tau_a} \right)
\]

where function \( a, b \) are defined in (5), they are depending of the states \( p_{im}, p_{ex}, p_c \).

The Lyapunov function is

\[
V = x^T Q^{-1}(x)x > 0, \quad V = (y_1, y_2, z)^T Q^{-1}(x)(y_1, y_2, z),
\]

where \( q_{ij} = q_{ij}(x) \), \( i = 1, n \) are state-depending components of matrix \( Q^{-1} \). The derivative of Lyapunov function is

\[
\dot{V} = x^T Q^{-1}(x)x + x^T Q^{-1}(x)\dot{x} + x^T \sum_{i=1}^{n} \left( \frac{\partial Q(x)}{\partial x_i} \right) x
\]

After decomposing the matrices we obtain

\[
\dot{V} =
2 \sum_{i=1}^{3} y_i^2 (q_{ii} f_{ii} + q_{2i} f_{2i} + q_{3i} (b_{31} f_{31} + b_{32} f_{32} + a_{3i}))+
+y_1 y_2 \left( \sum_{i=1}^{2} q_{i1} f_{i1} + q_{2i} + \sum_{i=1}^{2} b_{3i} f_{3i} + a_{31} \right)+
+y_1 \left( \sum_{i=1}^{2} q_{i1} f_{i1} + a_{31} \right)+
+y_2 \left( \sum_{i=1}^{2} q_{i2} f_{i2} + q_{3i} (b_{31} f_{31} + b_{32} f_{32} + a_{33}) \right)+
+y_2 \left( \sum_{i=1}^{2} q_{i2} f_{i2} + q_{3i} (b_{31} f_{31} + b_{32} f_{32} + a_{33}) \right)+
+y_2 \left( \sum_{i=1}^{2} q_{i2} f_{i2} + q_{3i} (b_{31} f_{31} + b_{32} f_{32} + a_{33}) \right)+
\]

(23)

It is easy to see that in the set

\[
\Omega = \{(y_1, y_2, z) : y_1, y_2, z > 0 \cup (y_1, y_2, z) : y_1, y_2, z < 0 \}
\]

the derivative can be sign-definite.

We define \( \tilde{a}_{31}(x) \), \( \tilde{a}_{32}(x) \), \( \tilde{b}_{31}(x) \), \( \tilde{b}_{32}(x) \) as maximal functions bounds relatively of the variables \( p_{im}, p_{ex}, p_c \). The majorization of these functions and their new \( (y_1, y_2, z) \) - depending forms are given in Appendix.

Substituting these bounds estimations into the equality (23) of Lyapunov function derivative, and collecting again into matrix form, we obtain the following inequality

\[
\dot{V} \leq 2 x^T Q(x) \left[ \tilde{A}(x) + \tilde{B}(x)F(x) \right] x + x^T \sum_{i=1}^{n} \left( \frac{\partial P(x)}{\partial x_i} \right) x < 0
\]

(24)

where

\[
\tilde{A}(x) = \begin{pmatrix}
0 & 0 & 0 \\
0 & 0 & 0 \\
\tilde{a}_{31} & \tilde{a}_{32} & 0
\end{pmatrix},
\tilde{B}(x) = \begin{pmatrix}
1 & 0 \\
0 & 1 \\
\tilde{b}_{31} & \tilde{b}_{32}
\end{pmatrix}
\]

Based on Theorem 1, we obtain the SOS condition in order to find the matrices \( Q(x), K(x) \)

\[
S_1 = Z^T Q^{-1}(x)Z \in SOS
\]

(25)

\[
S_2 = -Z^T \left[ Q^{-1}(x) \tilde{A}(x) + \tilde{A}(x) Q^{-1}(x) + \tilde{B}(x)K(x) + K^T(x) \tilde{B}(x) \right] Z + T_{min} \in SOS
\]

(26)

In order to obtain suitable SOS condition and SOS programming tool compatible on polynomial representation of matrix components we approximate the components by Tailor row, and add constraints to SOS conditions become feasible.
V. CONSTRUCTION OF CONTROL LAW. SOLVING IN SOS TOOLS

For regional stabilization, we will set state derivative bounds $m_1 = 1.9; m_2 = 0.01; m_3 = 80$. Assuming that $Q(x)$ and $K(x)$ have the following forms

$$Q(x) = Q_0(x) + Q_1(x) y_1 + Q_2(x) y_2 + Q_3(x) z$$

$$K(x) = K_0(x) + K_1(x) y_1 + K_2(x) y_2 + K_3(x) z$$

For monomial $(y_1, y_2, z)$ we solve condition (25)-(26) by SOS programming tool

$$Q_0 = 10^{-6} \times \begin{pmatrix} 460200 & 90890 & 32760 \\ 0.00009 & 0.000104 & 0.000023 \\ 32760 & 0.000023 & 0.00002 \end{pmatrix},$$

$$Q_1 = 10^{-6} \times \begin{pmatrix} -0.000235 & -0.00061 & -5854 \\ -0.000061 & 0.09851 & -0.05104 \\ -5854 & -0.05104 & 65030 \end{pmatrix},$$

$$Q_2 = 10^{-5} \times \begin{pmatrix} -0.5943 & -0.0098 & -1.704 \\ -0.0098 & 0.000057 & -504.3 \\ -1704 & -504.3 & 36.41 \end{pmatrix},$$

$$Q_3 = 10^{-5} \times \begin{pmatrix} 358.6 & 0.0051 & -0.000054 \\ 0.0051 & 504.31 & 0.00069 \\ -0.00005 & 0.000069 & 0.000011 \end{pmatrix},$$

$$K_0 = 10^{-3} \times \begin{pmatrix} 450740 & 0.07845 & 32.186 \\ 0.050232 & 0.55461 & -0.32467 \end{pmatrix},$$

$$K_1 = 10^{-4} \times \begin{pmatrix} -23.338 & 0.1511 & -10.21 \\ -0.1814 & 0.0034 & -0.8234 \end{pmatrix},$$

$$K_2 = 10^{-6} \times \begin{pmatrix} 15.111 & -0.0804 & -58963 \\ 0.34099 & 0.000049 & 6.437 \end{pmatrix},$$

$$K_3 = 10^{-5} \times \begin{pmatrix} -102.1 & -0.5896 & -0.000071 \\ -8.2344 & 0.6437 & -0.000012 \end{pmatrix},$$

$T_{\min} = 0.082$.

As a result, the stabilizing state feedback control law is $u(x) = K(x)Q^{-1}(x)x$.

The resulting controls $v_1$ and $v_2$ for the original system (1) can be obtained from (11) by applying $u(x)$ and by centering the controls at their setpoints.

The Fig.1 shows the phase portrait of the closed loop system.

VI. EXPERIMENTAL RESULTS

A. AMESim Virtual Test-Bench and Model validation

Since the control of diesel engines is mostly based on the maps, it cannot be optimized separately from the calibration process. A difference between analytical and empirical models should be taken account in order to achieve good control performance. Therefore the accuracy of dynamical model validation is an important task. The connection between the models is based on lookup-tables in which the measured data is stored. See also [18] for details.

The operating points for the designed SOS-based control strategy have been simulated on a four-cylinder diesel engine model, that was built in LMS AMESim professional software. This is a virtual testbed based on an intuitive graphical interface in which the system is displayed throughout the simulation process. The general performance parameters were taken from the static engine data and the reference values for the operating points were obtained from the experiments in order to evaluate the controller.

The control strategy has been tested on 4-cylinder diesel engine model which was built using co-simulation between AMESim and Matlab softwares. In Fig. 2 one can see the model validation for the intake and the exhaust pressure, compressor mass flow respectively. $P_{im} = 200000$ Pa, $P_{em} = 240000$ Pa, EGR valve$ = 30\%$, VGT vane opening$ = 0.3$, $W_c = 0.073$, $W_{egr} = 0.08$. The inputs of these maps are the
quantity of the fuel injected and the current engine speed. The maps are tuned according to low emissions and smoke.

As it was mentioned above in our approach it is necessary to invert the flow model to obtain the EGR valve position and the VGT vane position commands that are the control inputs to the model. Therefore the turbine and EGR mass flow rates and there dependence of the valves positions were determined and collected into the look-up table in order to use them for transformation. In Fig. 3 some simulation results are presented for EGR mass flow that depends of the valve. These experiments has been done in AMESim platform and cover large operating region. And by inversion of the flow control inputs the obtained EGR valve position and VGT vane position commands are passed to the engine.

### B. Operating Points and Control Simulation Results

The general performance parameters was taken from the static engine data and the reference values for the operating points were obtained from simulation experiments in AMESim software. To verify the effectiveness of the control strategies performances, they have been implemented in Simulink. The SOS-based control performance as well as PID controller have been tested in the same four operating points at $N = 2000$ (rpm) and 2500 (rpm); injection duration 1.2 (ms) and 1.8 (ms); valve positions $u_{egr} (%) = 20, 70, 10$ and $u_{vgt} (%) = 20, 30, 60$ (see Table II).

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Operating point 1</th>
<th>Operating point 2</th>
<th>Operating point 3</th>
<th>Operating point 4</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N$ (rpm)</td>
<td>2000</td>
<td>2000</td>
<td>2000</td>
<td>2500</td>
</tr>
<tr>
<td>Injection duration (ms)</td>
<td>1.2</td>
<td>1.8</td>
<td>1.2</td>
<td>1.8</td>
</tr>
<tr>
<td>$u_{egr} (%)$</td>
<td>20</td>
<td>70</td>
<td>70</td>
<td>10</td>
</tr>
<tr>
<td>$u_{vgt} (%)$</td>
<td>20</td>
<td>30</td>
<td>20</td>
<td>60</td>
</tr>
</tbody>
</table>

Fig. 7 and Fig. 8 show the intake and exhaust manifold pressure profiles that were to be achieved via the EGR and VGT control inputs regulation. In Fig.9 compressor mass flow rate demonstrates a smooth profile. Fig. 10 and Fig. 11 show satisfactory regulation without oscillations of the EGR flow rate and the turbine mass flow rate.

![Fig. 3. EGR mass flow dependence](image)

![Fig. 4. Desired and actual value of Intake Manifold Pressure. Dashed line - reference value, solid line - real value](image)

![Fig. 5. Desired and actual value of Exhaust Manifold Pressure. Dashed line - reference value, solid line - real value](image)

![Fig. 6. Dynamic of Compressor Mass Flow $W_c$. Dashed line - reference value, solid line - real value](image)

### C. Discussion and Comparison

The SOS-based controller has advantage and conventional structure, its response is uniform in different modes. To compare with conventional controllers such as PID which the response is given by Fig.9. The response is not uniform, it needs to be fine tuned separately for special modes, however it can cover a large operating region. The advantages and the effectiveness of SOS-based nonlinear control are concluded. The developed SOS-based nonlinear control approach gives a fast increase of air intake, graceful degradation in performances. Feasibility and performances of SOS-based control schemes as well as PID control can be firstly evaluated on the model and secondly validated experimentally on the engine.

The proposed approach will extend from trivial quadratic Lyapunov function case considered in different LPV ap-
proaches to more general polynomial Lyapunov function. Comparing to quadratic Lyapunov functions, this generalized Lyapunov function will help to construct the resulting nonlinear control law so that the design objective can be optimized with respect to a given cost functional for good performance. It improves control performance and it is able to expand a stability region of nonlinear systems.

We conclude that this model-based controller simplify the engine calibration. The simulations results depend on the accuracy of the engine model, so we compare the outputs of analytical and empirical models as it was shown above in order to achieve adequate and applicable control performance.

VII. ACKNOWLEDGMENTS

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VIII. CONCLUSIONS

We have developed a nonlinear robust controller based on SOS approach for the flows of the VGT-EGR systems. This nonlinear control approach gives a fast increase of variable performance and graceful degradation. Simulation experiments show the effectiveness of this approach. In the future work, a performance comparison with multiple model approaches, LPV and Takagi-Sugeno approaches and other types of nonlinear controller will be considered with application on real static maps based on a trade-off between maximal fuel economy and minimal emission and smoke. Also it is interesting to check different forms of polynomials that could affect the reliability, in order to achieve adequate and applicable control performance for different type of engines.

REFERENCES

APPENDIX

We find bounded functions $b_{31}(x) = b_{31l}(y_1,y_2,z)$, $b_{32}(x) = b_{32l}(y_1,y_2,z)$, $a_{31}(x) = a_{31l}(y_1,y_2,z)$, $a_{32}(x) = a_{32l}(y_1,y_2,z)$, such that $b_{31}(x,p_{im},p_{em},P_i) < b_{31l}(y_1,y_2,z)$; $b_{32}(x,p_{im},p_{em},P_i) < b_{32l}(y_1,y_2,z)$; $a_{31}(x,p_{im},p_{em},P_i) < a_{31l}(y_1,y_2,z)$; $a_{32}(x,p_{im},p_{em},P_i) < a_{32l}(y_1,y_2,z)$, relating variables $p_{im}, p_{em}, P_i$.  

1. We analyze the matrix component $b_{31}(x)$ and try to calculate the bounded function $\bar{b}_{31}(x)$. Noting that $b + a = \eta \frac{T_{em}}{\tau_{be}} (1-\eta_1(p_{im})+k_1\mu p_{im}^{-1}(y_1+W_{dc})) = \eta \frac{T_{em}}{\tau_{be}} (1-\eta_1(p_{im})+k_1\mu p_{im}^{-1}(y_1+W_{dc}))$. We obtain

$$b_{31}(x) = -k_2 = \mu p_{im}^{-1} - k_1 \frac{b}{\tau_{be}} = \mu p_{im}^{-1} - k_1 \frac{p_{im}^{-1}}{\eta_1(p_{im})}.$$ 

Taking into account that $p_{im} > (p_{im}^c)^{-\mu} - y_2 \mu$, obtain

$$b_{31}(x) = \eta \frac{T_{em}}{\tau_{be}} (1-\eta_1(p_{im})+k_1\mu p_{im}^{-1}(y_1+W_{dc})).$$

2. Consider the matrix component $b_{32}(x)$.

$$b_{32}(x) = -k_3 = \mu p_{im}^{-1} - \frac{p_{im}^{-1}}{\eta_1(p_{im})}.$$ 

Having divided the expression into 2 parts and analyzing both of them we obtain

$$b_{32}(x) = \mu \frac{T_{em} k_1}{\tau_{be}} \times$$

$$\times \frac{(y_1+W_{dc})k_1 \mu p_{im} - \eta \frac{T_{em}}{\tau_{be}} \eta_1(p_{im}) (p_{im}^{-\mu} - \eta_1(p_{im}) - 1)}{\mu \frac{T_{em} k_1}{\tau_{be}} (y_1+W_{dc})k_1 \mu + \frac{p_{im}^{-1}}{\eta_1(p_{im})}}.$$ 

Taking into account that $1 < p_{im}^{-\mu} < 2$ obtain

$$\bar{b}_{32}(x) = \frac{\mu \frac{T_{em} k_1}{\tau_{be}} \times}{\tau_{be}}$$

$$\times \frac{(y_1+W_{dc})k_1 \mu p_{im} - \eta \frac{T_{em}}{\tau_{be}} \eta_1(p_{im}) (p_{im}^{-\mu} - \eta_1(p_{im}) - 1)}{\mu \frac{T_{em} k_1}{\tau_{be}} (y_1+W_{dc})k_1 \mu + \frac{p_{im}^{-1}}{\eta_1(p_{im})}}.$$ 

3. Consider $a_{31}(x) = k_5 (b-1)$. Making analogy transformations we get

$$a_{31}(x) = \frac{\mu \frac{T_{em} k_1}{\tau_{be}} \times}{\tau_{be}}$$

$$\times \frac{(y_1+W_{dc})k_1 \mu p_{im} - \eta \frac{T_{em}}{\tau_{be}} \eta_1(p_{im}) (p_{im}^{-\mu} - \eta_1(p_{im}) - 1)}{\mu \frac{T_{em} k_1}{\tau_{be}} (y_1+W_{dc})k_1 \mu + \frac{p_{im}^{-1}}{\eta_1(p_{im})}}.$$ 

4. We find the bounded function of $a_{32}(x)$

$$a_{32}(x) = \frac{\mu \frac{T_{em} k_1}{\tau_{be}} \times}{\tau_{be}}$$

$$\times \frac{(y_1+W_{dc})k_1 \mu p_{im} - \eta \frac{T_{em}}{\tau_{be}} \eta_1(p_{im}) (p_{im}^{-\mu} - \eta_1(p_{im}) - 1)}{\mu \frac{T_{em} k_1}{\tau_{be}} (y_1+W_{dc})k_1 \mu + \frac{p_{im}^{-1}}{\eta_1(p_{im})}}.$$