Reverse Stripping Soft Decision Decoding for General Gaussian Fading Broadcast Channel

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I. INTRODUCTION

As one of the most intriguing multi-user communication problems, the broadcast channel (BC) has attracted the attention of researchers for many decades [1]-[7]. Of particular importance is the capacity region for the degraded BC for which superposition coding has been identified as a method that achieves the boundary of the capacity region [2], [3], [8]. For a two-user fading Gaussian degraded BC operating in low SNR regime, Berline and Tuninetti [9] have studied the design of low density parity check (LDPC) codes for 4-PAM signaling. Constrained by binary signaling to each receiver, the single-user capacity is upper bounded by 1 bit/symbol on both the in-phase and quadrature branches. In [10], a disjoint coding scheme based on bit-interleaved coded modulation (BICM) is proposed for degraded Gaussian channel, which also considers the low SNR regime but in the absence of fading. Nevertheless, when fading is present, it is shown in [11] that the capacity achieving input distribution is not Gaussian in general.

In contrast to the progress made in the degraded scenario, a single-letter characterization for the capacity region of a general fading Gaussian BC with neither degraded structure nor channel state information at transmitter (CSIT) remains unknown. Recently, Tse and Yates derived inner and outer bounds [12] within a constant gap of 6.386 b/s/Hz for all fading distributions. For the outer bound, they employ channel enhancement to induce a degraded BC and apply the well-known relationship between mutual information and minimum mean square error (MMSE) estimation [13]. For the inner bound, they proposed an achievability scheme based on a concatenation of binary expansion signaling (BES) and layered binary coding with hard decision decoding (HDD) at the receiver. However, as HDD discards the reliability information that can be exploited by soft-decision decoding (SDD), in this work we improve the achievability strategy presented in [7] by considering a layered low-density parity check (LDPC) coding framework. The rest of this paper is organized as follows. Section II introduces the system model for the BES signal encoded by layered LDPC codes over a general Gaussian fading BC. As an implementation for the chain rule of mutual information, Section III introduces a sub-optimal but low-complexity multistage SDD algorithm. To facilitate a practical implementation based on iterative decoding, Section IV gives an asymptotic analysis for the design of layered LDPC codes. Although the finite-length LDPC code design is a challenging issue, we show that the configuration of degree distribution can be formulated as a constrained optimization problem that takes into account the rule of rate assignment and the stability of message passing decoding. Numerical results are presented in Section V. Finally, Section VI concludes this paper.

II. SYSTEM MODEL

A. Proposed Transmitter and Encoder

The achievability scheme proposed in this paper is for broadcasting over a Gaussian fading channel, which is based on BES signaling and layered LDPC coding. Due to the symmetry of the in-phase and quadrature branches, we will consider the in-phase branch only unless otherwise stated.

To illustrate, Fig. 1 presents the block diagram for the proposed in-phase baseband transmitter with layered LDPC
B. Proposed Receiver and Decoder

As shown in Fig. 1, the communication channel from the transmitter to receiver \( i \) is associated with a non-negative channel gain \( S_i \) and Gaussian noise \( Z_i[t] \). The input signal of receiver \( i \) is given by

\[
\tilde{Y}_i[t] = \sqrt{S_i} \tilde{X}_i[t] + Z_i[t].
\]  

For convenience, we assume that \( S_i[t] \) and \( Z_i[t] \), \( i = 1, 2 \), are mutually independent processes and, individually, iid random sequences. In addition, we assume each \( Z_i[t] \) is a zero mean, unit variance Gaussian random variable.

The block diagram of receiver \( i \) with multi-stage LDPC decoder and BES demodulator is shown in Fig. 2. Since receiver \( i \) knows the instantaneous channel state information (CSI), \( S_i \) precisely, the AGC can normalize the received signal \( Y_i[t] \) by \( \sqrt{S_i}E \), yielding

\[
Y_i[t] = \frac{\tilde{Y}_i[t]}{\sqrt{S_i}E} = X[t] + Z_i[t],
\]  

where \( Z_i[t] = \tilde{Z}_i[t]/\sqrt{S_i}E \).

Given \( S_i = s \), \( Z_i[t] \) is a conditional Gaussian random variable with zero mean and variance \( 1/(sE) \). We note that the normalized form of (7) serves as the starting point for analysis of the communication system. We will often refer to \( X[t] \) as the transmitted signal and \( Y_i[t] \) as the receiver \( i \) input. This will simplify our subsequent presentation, although one must keep in mind that the channel fading is now hidden in the power fluctuations of the additive noise \( Z_i[t] \).

To avoid the complexity of maximum likelihood decoding, multi-stage decoding with successive cancelation is employed by each receiver. In the \( N \)-level LDPC-coded BES signal \( X[t] \), \( N_i = |N_i| \) signal levels are assigned to user \( i \) such that \( \{N_1, N_2\} \) partitions \( N \). Therefore, the BES signal (1) can be decomposed into

\[
X[t] = X_1[t] + X_2[t],
\]  

where the sub-constellation

\[
X_i[t] = \sum_{n \in N_i} X_n[t] 2^{-n},
\]  

employed for user \( i \) carries information encoded at rate

\[
R_i = \sum_{n \in N_i} r_n, \quad i = 1, 2.
\]  

Starting at level \( \kappa_1 \), receiver \( i \) collects channel observations regarding codeword \( [X_{\kappa_1}[1], X_{\kappa_2}[2], \ldots] \) and then derives a codeword estimate \( \hat{X}_{\kappa_1} = \hat{X}_{\kappa_1}[1], \hat{X}_{\kappa_1}[2], \ldots \). The contribution of the level \( \kappa_1 \) BES signal is then canceled, producing the first stage stripped signal

\[
Y_{i1}^1[t] = Y_i[t] - \hat{X}_{\kappa_1}[t] 2^{-\kappa_1}.
\]  

This serves as input to the stage 2 decoder for signal level \( \kappa_2 \) that yields the estimated codeword \( [\hat{X}_{\kappa_2}[1], \hat{X}_{\kappa_2}[2], \ldots] \). Similar to (9), the estimate for level \( \kappa_2 \) BES signal is subtracted from \( Y_{i1}^1[t] \), resulting in the stage 2 stripped signal \( Y_{i2}^2[t] \). This process repeats such that at stage \( n \) the level \( \kappa_n \) signal is decoded and stripped, producing the stripped signal

\[
Y_{i1}^n[t] = Y_{i1}^{n-1}[t] - \hat{X}_{\kappa_n}[t] 2^{-\kappa_n}, \quad n = 2, \ldots, \rho_i,
\]  

as input for the stage \( n + 1 \) decoder.

The sequence \( \kappa_1, \ldots, \kappa_{\rho_i} \) thus specifies a decoding schedule. While the decoding order can be arbitrarily chosen, we will focus on schedules arranged in the descending order

\[
\kappa_1 \succ \kappa_2 \succ \cdots \succ \kappa_{\rho_i}.
\]
In [7], such schedules are called reverse stripping, since the decoding order proceeds from the weakest to the strongest signal levels, which is the opposite of the traditional stripping of superposed signals.

In the ideal case, the codebooks of the $\rho_i$ layers are designed properly and the codeword estimates coincide with the true inputs for each time instant $t$, i.e.

$$\hat{X}_{\kappa_n}[t] = X_{\kappa_n}[t], \quad n = 1, 2, \ldots, \rho_i.$$  \hspace{1cm} (12)

In this event, for the $\rho_i$-stage decoding and demodulation for user $i$, there is no error propagation originating from LSB levels belonging to $S_i$, the set of signal levels decodable by receiver $i$. In contrast, the cross-interference from $S_i$, the complementary set of $S_i$, cannot be avoided. To evaluate the effect of cross-interference, let

$$A^m[t] = \sum_{l=1}^{m} X_l[t]2^{-l}, \quad 1 \leq m \leq N,$$  \hspace{1cm} (13)

denote the $m$-bit binary expansion of $X[t]$. Then, $Y_i^n[t]$, the outcome of stage $n$ stripping/decoding for user $i$, can be split into

$$Y_i^n[t] = A^{\kappa_n-1}[t] + U_i^n[t] + Z_i[t],$$  \hspace{1cm} (14)

where

$$U_i^n[t] = \sum_{j>\kappa_n, j \in \mathcal{D}_i} X_j[t]2^{-j}$$  \hspace{1cm} (15)

represents the LSB cross-interference undecodable by user $i$ up to level $\kappa_n \in \mathcal{N}_i$.

### III. SOFT DECISION DECODING

In [7], binary random codes are applied to each BES signal level at the transmitter side and hard decision decoding (HDD) is performed by each receiver. As a consequence, the physical channel between the receiver $i$ input $Y_i[t]$ and the BES demodulator output $\{\hat{X}_l[t], l \in \mathcal{N}_i\}$ is modeled by $N_i$ induced binary symmetric channels (BSC). The interference from the LSB’s in $\mathcal{N}_i$ is upper bounded by the crossover probabilities associated with a degraded BSC channel, in which the interference is a continuous uniform random variable. Nevertheless, we note that the reliability information of the binary decision is lost in such a HDD process [14].

As a remedy, we will employ soft-decision decoding (SDD) for each level, wherein the input messages are the bitwise log likelihood ratio (LLR) conditioned on the channel state. For modern LDPC channel codes, these LLRs provide sufficient statistic for a message passing decoder (MPD) [15]. For brevity, we drop the time index $[t]$, and the conditional LLR for $X_{\kappa_n}$ is given by

$$\Lambda_{\kappa_n}^{\kappa_n}(Y_i^n|S_i = s) = \log \left( \frac{P(Y_i^n|S_i = s, X_{\kappa_n} = 1)}{P(Y_i^n|S_i = s, X_{\kappa_n} = -1)} \right).$$  \hspace{1cm} (16)

where $\log(\cdot)$ denotes the nature log operator, $Y_i^n$ is the reversely stripped signal of stage-$n$ by receiver $i$, assuming the LSBs in $S_i$ have been detected correctly. A recursive definition for $Y_i^n$ has been given by (10). The conditional LLR in (16) is a soft decision regarding the unknown bit $X_{\kappa_n}$, which will be used as the input of level-$\kappa_n$ MPD.

Heuristically, the magnitude of $\Lambda_{\kappa_n}(Y_i^n|S_i = s)$ is proportional to the reliability of the decision regarding $X_{\kappa_n}$ [14]. As a result of the SDD metric in (16), we obtain an effective binary-input memoryless channel between $X_{\kappa_n}$ and $\Lambda_{\kappa_n}$. Furthermore, because of the truncation property of antipodal BES signaling [12], this channel is output symmetric, i.e.

$$P(\Lambda_{\kappa_n}|X_{\kappa_n} = 1) = P(-\Lambda_{\kappa_n}|X_{\kappa_n} = -1).$$  \hspace{1cm} (17)

Therefore, the concentration theorem for binary LDPC codes [15] can be extended to layered LDPC coding over parallel channels indexed by $\mathcal{N}_i$. To illustrate, Fig. 3 plots the symmetric transition probability for the effective binary input output symmetric channel regarding the MSB level for $N = 8$, $\mathcal{D}_1 = \{1, 3, 6\}$. In this example, we fix the channel gain to be 10 dB and evaluate the empirical PDFs of $P(\Lambda_1|X_1 = 1)$ and $P(\Lambda_1|X_1 = -1)$ for four possible scenarios of the binary pair $(X_3, X_6)$.

According to the definition of channel equivalence [15], we have the following observation.

**Lemma 1:** Using multi-stage soft-decision decoding with successive interference cancellation, the Gaussian fadding BC for user $i$ can be modeled as $N_i$ parallel memoryless channels with binary input $x \triangleq X_{\kappa_n}$ in $\{1, -1\}$ and LLR output $y \triangleq \Lambda_{\kappa_n}(Y_i^n|S_i = s) \in (-\infty, \infty)$. The transition probability of this channel is given by $P_{\kappa_n}(y|x)$, which satisfies the symmetric condition of $P_{\kappa_n}(y|x) = P_{\kappa_n}(-y|-x)$. \hspace{1cm} $\square$

In contrast to [7], wherein receiver $i$ treats all the signal levels of $\mathcal{N}_i$ as interference, here we cancel those signals in the set

$$\mathcal{N}_i^c = \{X_n|n \in \mathcal{N}_i \cap \mathcal{D}_i\}.$$  \hspace{1cm} (18)

Therefore, for an instantaneous channel state $S_i = s$, the rate achievable by receiver $i$ is a function of CSI and is subject to the upper bound

$$R_i \leq I(Y_i; X_i|\mathcal{N}_i^c, S_i).$$  \hspace{1cm} (19)

Since each level of sub-constellation is encoded independently, by applying the chain rule of mutual information [16]
to the right side of (19), we have

\[
I(Y_i; X_n|\mathcal{X}_i', S_i) = I(Y_i; \{X_n | n \in N_i\}|\mathcal{X}_i', S_i) = \sum_{i=1}^n I(Y_i'|X_{n_k} | S_i) \mathbb{I}_i(k_i)
\]

(20)

where \( \mathbb{I}_i(k_i) \) is a binary indicator of whether \( k \in N_i \). As a consequence, we can demonstrate higher achievable rates than the inner bound given by Theorem 5 of [7] for any specific channel instance.

**Lemma 2:** Using multi-stage soft-decision decoding with successive interference cancelation, the achievable rate region of the fading Gaussian BC \((S_1, S_2)\) includes all rate pairs \((R_1, R_2)\) satisfying

\[
R_i = \sum_{\kappa_n \in N_i} r_{\kappa_n} \leq \sum_{\kappa_n \in N_i} \mathcal{R}_{\kappa_i}^n, \quad i = 1, 2.
\]

(23)

where

\[ \mathcal{R}_{\kappa_i}^n = \int_{s=0}^{\infty} I(Y_i^n; X_{n_k} | S_i = s) f_{S_i}(s) ds. \]

(24)

That is, \( \mathcal{R}_{\kappa_i}^n \) gives a tight upper bound for the rate \( r_{\kappa_n} \) of level \( \kappa_n \in N_i \). In order to evaluate (24), let us consider the effective channel between \( X_{n_k} \) and its conditional LLR \( \Lambda_{\kappa_n}(Y_i^n | S_i = s) \) associated with the reverse stripping decoder given by Fig. 2. We assume channel state \( S_i \) is known and the decodable LSB levels \( X_{n_k}^{n-1} = \{X_n\}_{n=1}^{n-1} \) have been detected correctly. For simplicity of expression, we adopt the notation

\[
\pi_{n,i}^b(y|s) \triangleq f_{Y_i^n|S_i, X_{n_k}^i}(y|s, b)
\]

(25)

for the conditional density of \( Y_i^n \) given \( S_i = s \) and \( X_{n_k} = b \). As \( X_{n_k} \) is equiprobably \( \pm 1 \), independent of \( S_i \), the conditional marginal density of \( Y_i^n \) given \( S_i = s \) is

\[
f_{Y_i^n|S_i}(y|s) = \frac{1}{2} \pi_{n,i}^1(y|s) + \frac{1}{2} \pi_{n,i}^-1(y|s).
\]

(26)

It follows that for \( \kappa_n \in N_i \),

\[
I(Y_i^n; X_{n_k}|S_i = s) = h(Y_i^n|S_i = s) - h(Y_i^n|X_{n_k}, S_i = s) = \frac{1}{2} \int_0^{\infty} \sum_{b=-1}^1 \pi_{n,i}^b(y|s) \log \frac{2\pi_{n,i}^b(y|s)}{\pi_{n,i}^1(y|s) + \pi_{n,i}^-1(y|s)} dy.
\]

(27)

By substituting (28) into (24), we can obtain the upper bound of BES level \( \kappa_n \) for a given partition strategy \( \{N_1, N_2\} \) and decodable sets \( D_1 \) and \( D_2 \). Furthermore, the Bhattacharyya noise parameter (BNP) of the level-\( \kappa_n \) effective channel, \( X_{\kappa_n} \rightarrow \Lambda_{\kappa_n}(Y_i^n | S_i = s) \), is given by

\[
\mathbb{B}_{\kappa_n}(s) = \int_{R} \exp(-|m|/2) f(m) dm,
\]

(29)

where \( m = \Lambda_{\kappa_n}(Y_i^n | s) \) is the conditional LLR defined by (16) and \( f(m) \) is its PDF. Given the PDF \( f_{S_i}(s) \), the average BNP for level \( \kappa_n \) of fading channel \( i \) is

\[
\mathbb{B}_{\kappa_i}^n = \int_0^{\infty} \mathbb{B}_{\kappa_n}(s) f_{S_i}(s) ds.
\]

(30)

IV. LAYERED LDPC CODE DESIGN

Assume a layered LDPC encoder is employed at the transmitter, and a multi-stage decoder with reverse stripping is employed at the receiver. Following [15], the degree distribution polynomials for level-l LDPC code are given by

\[
\lambda_l(x) = \sum_i \lambda_{l,i} x^{i-1}
\]

(31a)

\[
\rho_l(x) = \sum_k \rho_{l,k} x^{k-1},
\]

(31b)

where \( \lambda_{l,i} \) and \( \rho_{l,k} \) denote the fraction of edges with variable node (VND) degree \( i \) and check node (CND) degree \( k \), respectively. By definition, we have

\[
\sum_i \lambda_{l,i} = 1,
\]

(32a)

\[
\sum_k \rho_{l,k} = 1.
\]

(32b)

Our goal is to find degree distributions for \( \{\lambda_l(x), \rho_l(x)\} \), which result in codes with vanishing BER as the block length and the number of iterations for message passing decoding [15] increase. We analyze the asymptotic performance of the LDPC \( \{n, \lambda_1(x), \rho_1(x)\} \) ensemble for \( n \rightarrow \infty \) and we restrict our attention to the cycle-free case. Generally, the stability of an iterative decoder for LDPC codes can be identified by the BNP, regardless of the details of channel types [15]. In light of this, if level \( l \) is assigned to user \( i \), then the stability condition [7] for the degree pairs \( \{\lambda_l(x), \rho_l(x)\} \) under message-passing decoding is given by

\[
\frac{1}{\lambda_0(\rho_0)} \frac{1}{\lambda_0(\rho_0)} > \mathbb{B}_{l}^n.
\]

(33)

and the design rate is constrained by

\[
1 - \frac{1}{\sum_k \rho_{l,k}} \leq \mathcal{R}_i.
\]

(34)

Due to page limitation, more details about the LDPC design code are given in [17].

V. NUMERICAL RESULTS

In order to illustrate the gains in the achievable rates enabled by SDD, we use the inner and outer bounds derived for intermittent AWGN broadcast channel [7] as our baseline, wherein user \( i \) has channel state \( S_i \) described by the distribution

\[

\mathcal{F}_{S_i}(s) = \begin{cases} 1 & s < 0, \\ \frac{p_i}{s_i} & 0 \leq s \leq s_i, \\ 0 & s > s_i. \end{cases}
\]

(35)

Following [7], we refer to \( p_i \) as the channel activity factor and \( s_i \) as the maximum SNR. The ergodic capacity for the intermittent complex AWGN channel modeled by (35) is

\[
C_i = p_i \log(1 + s_i).
\]

(36)

In all such examples, we assume without loss of generality that \( s_{i}^2 \leq s_i^2 \) and \( p_i \geq p_2 \).

To illustrate, Fig. 4(a) considers the case for degraded Gaussian BC with \( p_1 = p_2 = 1, s_1^2 = 20\text{dB}, s_2^2 = 10\text{dB} \).
and Fig. 4(b) considers another degraded Gaussian BC for $p_1 = p_2 = 1$, $s_1^2 = 50$ dB, $s_2^2 = 20$ dB. Each plot compares the outer bound, the capacity, the inner bounds with soft/hard decision decoding, and with reverse stripping (RS) or without.

Next we consider a corresponding pair of BC in which user 2 has the same AWGN channel with $p_2 = 1$ but user 1 now has a channel activity factor $p_1 < 1$ such that users 1 and 2 have equal ergodic capacities $C_1 = C_2$. For this case, numerical comparisons of the inner and outer bounds are given in Fig. 5(a-b). In particular, user 1 has activity factor (a) $p_1 \approx 0.52$ and (b) $p_1 = 0.4$. Each plot also compares the outer bound, the capacity, the inner bounds with soft/hard decision decoding and with/without reverse stripping. Still, we can see that SDD can boost the inner bounds achieved by HDD, with or without reverse stripping. For the intermittent fading channel of Fig. 5, SDD with reverse stripping appears to be within one bit of the capacity outer bound in [7] and for much of the boundary, the gap is considerably smaller.

VI. Conclusion

In this work, we extended the BES-HDD achievability strategy of Tse and Yates to BES-SSD. We observed that by invoking layered LDPC coding at the transmitter and successive SDD at receivers, a considerable improvement in achievable rates can be accomplished. Based on the requirement for stability of message passing decoding as well as the rule for code rates allocation, we derived the necessary conditions for the degree distribution of capacity-achieving LDPC codes. The construction of finite-length LDPC codes remains an important and challenging issue due to practical constraints regarding decoding threshold, latency and error floor.

REFERENCES


Fig. 4. AWGN broadcast channel rate regions: (a) users 1 and 2 have SNRs 20 dB and 10 dB. (b) users 1 and 2 have SNRs 50 dB and 20 dB. Each plot compares the outer bound, the capacity, and the inner bounds with hard or soft decision decoding, and with reverse stripping (RS) or without.

Fig. 5. AWGN intermittent broadcast channel rate regions: (a) users 1 and 2 have SNRs 20 dB and 10 dB, $p_1 \approx 0.52$, $p_2 = 1$. (b) users 1 and 2 have SNRs 50 dB and 20 dB, $p_1 = 0.4$, $p_2 = 1$. 

50 dB, 8;