Neural Network Dynamic Programming Constrained Control of Distributed Parameter Systems Governed by Parabolic Partial Differential Equations with Application to Diffusion-Reaction Processes

Behzad Talaei\(^1\), Hao Xu\(^2\) and S. Jagannathan\(^1\)

I. Abstract

In this paper, a novel neural network (NN) adaptive dynamic programming (ADP) control scheme for distributed parameter systems (DPS) governed by parabolic partial differential equations (PDE) is introduced in the presence of control constraints and unknown system dynamics. First, Galerkin method is utilized to develop a relevant reduced order system which captures the dominant dynamics of the DPS. Subsequently, a novel control scheme is proposed over finite horizon by using NN ADP. To relax the requirement of system dynamics, a novel NN identifier is developed. Moreover, a second NN is proposed to estimate online the time-varying non-quadratic value function from the Hamilton-Jacobi-Bellman (HJB) equation. Subsequently, by using the identifier and the value function estimator, the optimal control input that inherently lies in actuation limits is obtained. A local uniform ultimate boundedness (UUB) of the closed-loop system is verified by using standard Lyapunov theory. The performance of proposed control scheme and effects of its design parameters are successfully verified by simulation on a diffusion reaction process.

II. Introduction

Partial differential equations (PDE) arise in almost all real-world engineering problems to describe distributed parameter systems (DPS). Although in many applications, a model described by ordinary differential equations (ODE) may suffice, many other industrial processes require analysis and control of PDE. In addition, since these systems are distributed in space, regulation of DPS needs considerable control effort. As a consequence, such DPS require control systems that also consider minimizing the control effort.

Extensive research on control of DPS has been reported in the literature [1]–[8]. Among practical methods, the control of PDE systems by using backstepping approach [1]–[3] and advanced model reduction techniques [4]–[8] has attracted considerable attention. The first approach relaxes the requirement on in-domain actuators or sensors and applies the feedback control as a boundary condition input whereas the latter derives a finite dimensional approximated model that accurately reproduces the dominant dynamics of the original DPS and designs a feedback controller that is robust to uncertainties in this finite dimensional domain.

Though nonlinear [2] or adaptive [3] control approach appeared promising, optimal control of DPS by using boundary excitation without any model reduction still remains a challenging problem. However, if the model reduction technique is used, i.e. dominant finite set of states can be estimated and the key inputs inside the system domain can be controlled by taking advantage of state of the art adaptive dynamic programming (ADP) schemes for finite dimensional systems [9].

Therefore, in this paper, the ADP based control of DPS by using model reduction techniques is addressed. In particular, a NN based ADP control scheme of DPS governed by semi-linear parabolic PDE with control input constraints is proposed. Unlike [1]–[8], the system dynamics are considered unknown. Second, in contrast to [5]–[7] which uses nonlinear optimization schemes, the proposed control is solved forward in time without using iterations for online implementation. Moreover, Lyapunov proof in this work demonstrates the stability of the closed-loop system. In addition, unlike [4], the optimal policy minimizes the time-varying value function over finite horizon. Finally, unlike [4] and [7], this effort considers the control constraints due to actuator limits. Neglecting such constraints can lead to significant performance deterioration and closed-loop instability.

Thus, in this paper first the Galerkin approximation technique is utilized to obtain a finite dimensional approximated representation of the system for the purpose of controller development. Subsequently, a NN identifier is proposed to learn the system dynamics and a second NN is introduced to learn the time-varying value function over finite horizon in a forward-in-time manner. Thereafter, the estimated optimal control input is derived by using value function estimator and identified system dynamics.

It is important to note that instead of value or policy iterations [11] proposed method utilizes historical information of the past value function estimates and its error, and the value function and control input are updated once a sampling instant consistent with the conventional adaptive and real time control. In order to incorporate control constraints in control design, a non-quadratic cost function is considered. Eventually, standard Lyapunov analysis is utilized to demonstrate the closed-loop stability of the proposed scheme. Simulation results confirm the effectiveness of the proposed
scheme on a diffusion-reaction process.

The rest of the paper is organized as follows. In Section III, the class of PDE systems under consideration is described and the finite dimensional system is derived by using Galerkin approximation technique. In Section IV, the NN ADP controller for the uncertain finite dimensional system is designed. In Section V, the overall stability of proposed method is addressed. Section VI demonstrates the simulation results and Section VII provides the conclusions of the paper.

III. PRELIMINARIES

A. Class of DPS

We consider the class of DPS described by semi-linear parabolic PDE systems of the form:

\[
\begin{align*}
\frac{\partial \tilde{x}}{\partial t} &= A \frac{\partial \tilde{x}}{\partial z} + B \frac{\partial^2 \tilde{x}}{\partial z^2} + \sum_{i=1}^{m} b_i(z) u_i + f(\tilde{x}), \\
D_1 \tilde{x}(\alpha, t) + E_1 \frac{\partial \tilde{x}}{\partial z}(\alpha, t) &= R_1, \\
D_2 \tilde{x}(\beta, t) + E_2 \frac{\partial \tilde{x}}{\partial z}(\beta, t) &= R_2,
\end{align*}
\]

with the initial condition:

\[
\tilde{x}(z, 0) = \tilde{z}_{0}(z),
\]

where \(\tilde{x}(z, t) \in H^n_2\) represents the state vector with \(H^n_2\) being the \(n\)-dimensional Sobolev space of order 2, \(z \in [\alpha, \beta] \subset \mathbb{R}\) being the spatial coordinate, and \(t \in [0, \infty)\) represents the time. \(u_i \in [u_{i,\min}, u_{i,\max}]\) denotes the \(i\)th control input, \(\partial \tilde{x}/\partial z, \partial^2 \tilde{x}/\partial z^2\) represent the first and second-order spatial derivatives of \(\tilde{x}\). \(f(\tilde{x}) \in C^m\) is an unknown nonlinear vector function with \(C\) being the space of continuous functions, \(A, B, D_1, E_1, D_2, E_2\) are constant matrices, \(R_1, R_2\) are column vectors, and \(\tilde{z}_{0}(z)\) being the initial condition. The term \(b_i(z), 1 \leq i \leq m\) is a known smooth vector function of \(z\) which describes how the control action \(u_i(t)\) is distributed in the interval [\(\alpha, \beta]\).

B. Galerkin Method \[8\]

To make the problem tractable, an assumption is made such that the eigen spectrum of DPS (1) can be partitioned into a finite dimensional part consisting of \(m\) slow eigenvalues and a stable infinite complement containing the remaining fast eigenvalues, and that the separation between the slow and fast eigenvalues of the DPS is significant. Under this assumption, we apply standard Galerkin’s method \[10\] to the system of equations given by (1) in order to derive an approximate finite dimensional system. To this end, we first rewrite equation (1) as

\[
\tilde{x} = \tilde{A} \tilde{x} + \tilde{B} u + f(\tilde{x}), \quad \tilde{x}(0) = \tilde{x}_{0},
\]

where \(x(t) = \tilde{x}(z, t), \tilde{A} \tilde{x} = \frac{\partial \tilde{x}}{\partial z}, \tilde{B} \frac{\partial^2 \tilde{x}}{\partial z^2} + \sum_{i=1}^{m} b_i(z) u_i\) and \((\ldots)\) being the conventional \(L^2\) space inner product. Defining \(H_f = \text{span}\{\phi_1, \phi_2, \ldots, \phi_m\}\) as the slow modal subspace and \(H_I = \text{span}\{\phi_{m+1}, \phi_{m+2}, \ldots\}\) as the fast and stable modal subspace, and taking the inner product with the adjoint eigen-functions \(\phi_i^*, i \geq 1\), the state \(x\) of the system (2) can be decomposed as

\[
x = x_s + x_f = P_s x + P_f x,
\]

where \(P_s\) and \(P_f\) are corresponding inner product operators of spaces \(H_s\) and \(H_I\), and \(x_s\) and \(x_f\) are state vectors of these modal subspaces respectively. Then equation (2) can be equivalently written as

\[
\begin{align*}
\frac{dx_s}{dt} &= \tilde{A}_s x_s + \tilde{B}_s u + f_s(x_s, x_f), \\
\frac{dx_f}{dt} &= \tilde{A}_f x_f + \tilde{B}_f u + f_f(x_s, x_f), \\
x_s(0) &= P_s x(0), \quad x_f(0) = P_f x(0),
\end{align*}
\]

where \(\tilde{A}_s = P_s \tilde{A}, \tilde{B}_s = P_s \tilde{B}, \tilde{f}_s = P_s f, \tilde{A}_f = P_f \tilde{A}, \tilde{B}_f = P_f \tilde{B}, \tilde{f}_f = P_f f\) and the partial derivative notation in \(\partial x_f/\partial t\) is used to denote that the state \(x_f\) belongs to an infinite-dimensional space. In the above system, \(\tilde{A}_s\) is a diagonal matrix of dimension \(m \times m\) of the form \(\tilde{A}_s = \text{diag}\{\lambda_j\}\), \(f_s(x_s, x_f) \in C^m\) and \(f_f(x_s, x_f)\) is an infinite dimensional Lipschitz vector function which is exponentially stable. Neglecting the fast and stable infinite-dimensional \(x_f\)-subsystem in (4), the \(m\)-dimensional slow system is obtained as

\[
\frac{d\bar{x}_s}{dt} = \tilde{A}_s \bar{x}_s + \tilde{B}_s u + f_s(\bar{x}_s, 0),
\]

where \(\bar{x}_s \in C^m\) is the state vector associated with the finite-dimensional system.

IV. ADP CONTROL OF DPS Modeled by PARABOLIC PDE USING NEURAL NETWORKS

In order to simplify notation in (5), let the finite dimensional slow system be expressed in the following form as

\[
\dot{x}_s = \tilde{f}_s(\bar{x}_s) + \tilde{g}_s(\bar{x}_s)u,
\]

where \(\tilde{f}_s(\bar{x}_s) = \tilde{A}_s \bar{x}_s + \tilde{f}_s(\bar{x}_s, 0)\) and \(\tilde{g}_s(\bar{x}_s) = \tilde{B}_s\) and \(\tilde{f}_s(\bar{x}_s) : C^m \rightarrow C^m, \tilde{g}_s(\bar{x}_s) : C^m \rightarrow C^m\) represent smooth nonlinear functions which are considered unknown. However, it is assumed that the state vector is available and the input matrix \(\tilde{g}_s(\bar{x}_s)\) is considered bounded such that \(\|\tilde{g}_s(\bar{x}_s)\| \leq g_{\max}\). Without loss of generality, the system is considered controllable within control constraint in the sense that there exists a continuous control policy \(u\), \(u_i \in [u_{i,\min}, u_{i,\max}]\), that stabilizes the system, with origin being a unique equilibrium point \(\bar{x}_s\) on a set \(\Omega \subset \mathbb{R}^m\). The objective of the control design then is to determine a feedback control policy over finite-horizon that minimizes the following value function

\[
V(\bar{x}_s, t_0) = \psi(\bar{x}_s(t_f), t_f) + \int_{t_0}^{t_f} r(\bar{x}_s, u, t) dt,
\]

subject to the system dynamics (6). The function \(\psi(\cdot)\) penalizes the state \(\bar{x}_s(t_f)\) at the terminal time \(t_f\) and \(r(\bar{x}_s, u, t)\) penalizes the state \(\bar{x}_s\) and the control input \(u\) from \(t_0\) to \(t_f\). The cost-to-go function \(r(\bar{x}_s, u, t)\) generally takes the form as

\[
r(\bar{x}_s, u, t) = Q(\bar{x}_s, t) + 2 \int_{0}^{\alpha} t \rho(v) - T R dv
\]
a positive definite function of the state and time, \( R \) is a symmetric positive definite matrix with appropriate dimension and \( \rho^{-1}(.) \) is inverse of a saturation vector function that incorporates the control constraints. The nonquadratic function is used to represent the control effort outside the actuator limits as extremely expensive. As a rule of thumb, a hyperbolic tangent sigmoid transfer function within area of operation of the actuators can be chosen as \( \rho(.) \). Under the assumption that \( V(x,t) \in C^1 \), an infinitesimal equivalent to (7) is given by \[9\]

\[- \frac{\partial V(\bar{x}_t,t)}{\partial t} = r(\bar{x}_t,u,t) + \frac{\partial V(\bar{x}_t,t)}{\partial \bar{x}_t}(\bar{f}_s(\bar{x}_t) + \bar{g}_s(\bar{x}_t))u.\]  

By setting \( t_0 = t_f \), the terminal condition for the value function is given by

\[V(\bar{x}_f,t_f) = \psi(\bar{x}_f(t_f),t_f),\]  

Next define the Hamiltonian as

\[H(\bar{x}_t,u,t) = V_t + Q(\bar{x}_t,t) + 2 \int_0^u f^T(\bar{v})Rd\bar{v},\]  

\[+ V^s_T(\bar{f}_s(\bar{x}_t) + \bar{g}_s(\bar{x}_t))u,\]  

where \( V_t = \frac{\partial V(\bar{x}_t,t)}{\partial t} \) and \( V_{\bar{x}_t} = \frac{\partial V(\bar{x}_t,t)}{\partial \bar{x}_t} \). Note that since the finite-horizon value function (7) depends upon time, equation (10) has the time dependency term \( V_t \) in contrast to infinite-horizon case where it is omitted in the Hamiltonian. Therefore, the optimal control policy is obtained by using the well-known stationary condition \( \partial H(\bar{x}_t,u,t)/\partial u = 0 \), which yields

\[u^*(\bar{x}_t,t) = -\rho\left(\frac{1}{2}R^{-1}\bar{g}_s^T(\bar{x}_t)V^s_{\bar{x}_t}\right),\]  

with \( V^*(\bar{x}_t,t) \) being the optimal time-varying value function. Note that using the nonquadratic value function (7) results in an optimal control policy that lies within actuator limits. This is completely different from standard non-constraint optimal control [12] where the optimal policy may burden huge initial or maintenance costs for actuators.

Substituting (11) into (8) yields the non-quadratic time-varying HJB equation given by

\[V_t^s + V^s_T(\bar{f}(\bar{x}_t)) + Q(\bar{x}_t,t) + 2 \int_0^u f^T(\bar{v})Rd\bar{v},\]

\[+ V^T(\bar{x})^Tf(\bar{x}_t) - V_T^s(\bar{g}(\bar{x}_t))V^s(\bar{x}_t))\rho\left(\frac{1}{2}R^{-1}\bar{g}_s^T(\bar{x}_t)V^s_{\bar{x}_t}\right) = 0.\]  

A. NN-identifier design

Normally, the system dynamics are required for developing control of nonlinear systems in affine form. However, since system dynamics are generally uncertain, a NN-identifier is introduced. Recalling the NN universal function property, the nonlinear continuous system can be represented over a compact set \( \Omega \) as

\[\bar{f}_s(\bar{x}_t) = W_f^s \sigma_f(\bar{x}_t) + \varepsilon_f,\]

\[\bar{g}_s(\bar{x}_t) = W_g^s \sigma_g(\bar{x}_t) + \varepsilon_g,\]  

where \( W_f \in \mathbb{R}^{m \times l} \) and \( W_g \in \mathbb{R}^{m \times l} \) represent the target weight matrices, \( \sigma_f : C^m \rightarrow C^l \), \( \sigma_g : C^m \rightarrow C^l \) denote activation functions and \( \varepsilon_f \in C^{m \times 1} \), \( \varepsilon_g \in C^{m \times 1} \) denote NN reconstruction errors respectively.

Then, the nonlinear continuous-time system (6) in affine form can be represented by using (13) as

\[\dot{\bar{x}}_t = \bar{f}_s(\bar{x}_t) + \bar{g}_s(\bar{x}_t)u = W_f^T \sigma_f(\bar{x}_t) + W_g^T \sigma_g(\bar{x}_t)u + \varepsilon_f + \varepsilon_gu = W_f^T \sigma_f(\bar{x}_t)\bar{u} + \varepsilon_t,\]  

where \( W_f = [W_f^T, W_g^T]^T \in \mathbb{R}^{2l \times n} \), \( \sigma_f(\bar{x}_t), \sigma_g(\bar{x}_t) \) are target NN identifier weight and activation function, \( \bar{u} = [1 u^T]^T \in C^{m+1} \) is augmented control input and \( \varepsilon_t = \varepsilon_f + \varepsilon_gu \) is the NN identifier reconstruction error. Since activation functions for the NN identifier \( \sigma_f(\cdot), \sigma_g(\cdot) \) are known, the nonlinear system dynamics can be approximated while the NN identifier weight matrix \( W_f \) is tuned. Next, the update law for the online NN identifier will be introduced.

Consider the following state estimator or identifier given by

\[\dot{\hat{x}}_t = \hat{W}_f^T \sigma_f(\hat{x}_t)\hat{u} + K\bar{x}_t,\]  

\[\hat{W}_f \in C^{2l \times 1} \] is the estimated NN identifier weight matrix, \( \hat{x}_t \in C^m \) is the state estimation error and \( K \) is the design parameter for maintaining the stability of the NN identifier. Recalling equations (13) and (14), the dynamics of state estimation or identification error can be expressed as

\[\dot{\hat{x}}_t = \hat{x}_t - \hat{x}_t = \hat{W}_f^T \sigma_f(\hat{x}_t)\hat{u} + \varepsilon_t - K\bar{x}_t.\]  

In order to force the actual NN identifier weight matrix close to its target within a finite time, the update law for \( \hat{W}_f \) can be selected as

\[\dot{\hat{W}}_f = -\alpha_f\hat{W}_f + \sigma_f(\bar{x}_t)\bar{u}\bar{x}_t,\]  

where \( \alpha_f \) is tuning parameter of the NN identifier satisfying \( \alpha_f > 0 \). Since \( \hat{W}_f = -\hat{W}_f \), the dynamics of NN identifier weight estimation error can now be represented as

\[\dot{\hat{W}}_f = \alpha_f\hat{W}_f - \sigma_f(\bar{x}_t)\bar{u}\bar{x}_t.\]  

Next, the development of controller is introduced.

B. NN ADP control design

Traditional adaptive critic based schemes utilize two NNs, one for the value function, referred to as critic network, and the second for the control input, referred to as action network, in order to obtain admissible control inputs. However, in this paper, the adaptive critic scheme is realized by using only a single NN in an online fashion. To guarantee the terminal constraint while maintaining the system stability, an error term corresponding to the terminal constraint is introduced and minimized.

According to the universal approximation property of NN, the time-varying value function, \( V^*(x,t) \), can be expressed by using a NN with time-varying activation function on a compact set \( \Omega \) in the form

\[V^*(\bar{x}_t,t) = W_f^T \phi(\bar{x}_t,t - t_f) + \varepsilon_V(\bar{x}_t,t),\]  

where
where $W_t \in \mathbb{R}^{r \times 1}$ is the target NN weight vector with $r$ being the number of hidden-layer neurons, $\phi(\tilde{x}_s,t-t_f): [0, \infty) \times C^r \rightarrow \mathbb{R}$ is bounded time-dependent activation function, $\epsilon_V(\tilde{x}_s,t)$ is the NN reconstruction error. The target NN weights $W_t$ and reconstruction error $\epsilon_V(\tilde{x}_s,t)$ are assumed to be bounded above such that $\|W_t\| \leq W_M$ and $\|\epsilon_V(\tilde{x}_s,t)\| \leq \epsilon_M$, where $W_M$ and $\epsilon_M$ are positive constants. In addition, it is assumed that the gradient of the NN reconstruction error with respect to $\tilde{x}_s$ is bounded above such that $\|\nabla_{\tilde{x}_s} \epsilon_V(\tilde{x}_s,t)\| \leq \epsilon_M$, where $\epsilon_M$ is also a positive constant. The quantities $\phi(\tilde{x}_s(t_f),0)$ and $\epsilon_V(\tilde{x}_s,t_f)$ have the same meaning but corresponds to the terminal time and state. Note that the NN representation (19) enjoys time-dependency, which is essential to approximate the time-varying nature of the value function $V(\tilde{x}_s,t)$.

From equation (19), the partial derivatives of $V(\tilde{x}_s,t)$ with respect to $\tilde{x}_s$ and $t$ are given by

$$
V_{\tilde{x}_s} = \nabla_{\tilde{x}_s} \phi^T(\tilde{x}_s,t_f-t)W_t + \nabla_{\tilde{x}_s} \epsilon_V(\tilde{x}_s,t),
$$

$$
V_t = \nabla_{\tilde{x}_s} \phi^T(\tilde{x}_s,t_f-t)W_t + \nabla_{\tilde{x}_s} \epsilon_V(\tilde{x}_s,t),
$$

(20)

where $\nabla_{\tilde{x}_s} \phi(\tilde{x}_s,t_f-t) = \frac{\partial \phi(\tilde{x}_s,t_f-t)}{\partial \tilde{x}_s}$, $\nabla_{\tilde{x}_s} \epsilon_V(\tilde{x}_s,t)$ is the NN reconstruction error. The target NN weights estimation errors for the NN identifier and controller can be given by (17) and (27) respectively while the estimated HJB error boundedness is to minimize the approximated Hamiltonian (23) and the terminal constraint error will be satisfied approximately. Under the update laws (17) and (27) and control input (24), it will be shown in Section V-A that the state $\|\tilde{x}_s\|$ and the NN weights estimation errors for the NN identifier and the controller $\|\tilde{W}_f\|$ and $\|\tilde{W}_\eta\|$ will be UUB.

V. STABILITY ANALYSIS

A. Stability of slow finite dimensional part

In this section, the overall stability of the closed-loop finite dimensional system is analyzed. It will be shown that all the signals are UUB. Due to the time-varying nature of the value function and the terminal constraint terms of the Hamiltonian (7) and the truly optimal control policy $u^*(\tilde{x}_s,t)$ (11). Provided that the dynamics are known, the state vector $\tilde{x}_s$ is asymptotically stable.

Proof: Omitted due to space constraints.

Next, the overall stability of proposed NN ADP control scheme is demonstrated.

Theorem 1 (NN-based controller HJB error boundedness and the stability of finite dimensional system): Given the nonlinear system (6) with the target HJB equation (12), let the NNs update law for the identifier and the controller be given by (17) and (27) respectively while the estimated control input be given as (24). Then, the state error $\|\tilde{x}_s\|$ and the NN weights estimation errors for the NN identifier and the controller $\|\tilde{W}_f\|$ and $\|\tilde{W}_\eta\|$ respectively are UUB.

Proof: Omitted due to space constraints.

B. Stability analysis of infinite dimensional system

From the stability result in part V-A, it is clear that the state $\tilde{x}_s$ is bounded. The original closed-loop DPS can be written in the following form [14] as

$$
\frac{dx_s}{dt} = \tilde{A}_s x_s + \tilde{B} u + f_s(x_s,0) + [f(\tilde{x}_s,\tilde{x}_f) - f_s(x_s,0)],
$$

$$
\mu \frac{\partial x_f}{\partial t} = \tilde{A}_f x_f + \mu \tilde{f}(x_s,\tilde{x}_f),
$$

(28)
where \( \tilde{A}_{f\mu} \) is an unbounded differential operator defined as \( \tilde{A}_{f\mu} = \mu \tilde{A}_{f} \), \( \mu = [\text{Re}(\lambda_{1})],[\text{Re}(\lambda_{m+1})] \) and \( \tilde{A}_{f}(x_{f},x_{f}) = -\beta_{t} + f_{f}(x_{f},x_{f}) \). Note that \( f_{f}(x_{f},x_{f}) \) is independent of \( \mu \). Assuming \( \mu \) is a small positive number less than one, the fast time-scale \( \tau = t/\mu \) is introduced and setting \( \mu = 0 \), we obtain the following infinite-dimensional fast subsystem from (28) as

\[
\frac{d\tilde{x}_{f}(\tau)}{d\tau} = \tilde{A}_{f\mu}\tilde{x}_{f},
\]

where the bar symbol in \( \tilde{x}_{f} \) denotes that the state \( \tilde{x}_{f} \) is associated with the approximation of the fast \( x_{f} \)-subsystem. From the fact that \( \text{Re}(\lambda_{m+1}) < 0 \) and the definition of \( \mu \), we have that the above system is globally exponentially stable. Therefore there exists real numbers \( k_{1} \geq 1, \ a_{1} > 0 \) such that

\[
\|\tilde{x}_{f}(\tau)\|_{2} \leq k_{1}\|\tilde{x}_{f}(0)\|e^{-a_{1}t} \quad \forall t \geq 0.
\]

Assuming \( \mu \approx 0 \) in the system of (28) and using the fact that the operator \( \tilde{A}_{f\mu} \) is invertible, we have that \( \tilde{x}_{f} \approx 0 \). The finite dimensional closed-loop slow system therefore reduces to the one analyzed in section V-A where we have already shown that it is locally UUB. Therefore, given any \( \delta_{f} \) where \( \|x_{f}(0)\|_{2} \leq \delta_{f} \) and provided that \( x_{0}(0) \in \Omega_{b} \), where \( \Omega_{b} = \{x_{f} \in C^{\infty}[x_{f}] \leq \delta_{b}\} \) and \( \Omega_{b} \subset \Omega \), there exists \( \mu(1) > 0 \) such that if \( \mu \in (0, \mu(1)] \), then, for all \( t \geq 0 \), the states of the closed-loop system satisfy

\[
\|\tilde{x}_{f}(t)\|_{2} \leq k_{2}\|x_{0}(0)\|, \\
\|x_{f}(t)\|_{2} \leq k_{1}\|x_{0}(0)\|e^{-a_{1}(t/\mu)} + d,
\]

where the offset \( d \) depends on \( \mu \). The above inequalities imply that assuming the parameter \( \mu \) is sufficiently small, the trajectories of the closed-loop system will be bounded. Furthermore, as \( t \) increases, these trajectories will be ultimately bounded with an ultimate bound depending on \( d \).

Remark 3: The assumption that \( \mu \) is sufficiently small, can be interpreted as the neglectable stable infinite dimensional part has a negligible effect on the overall system dynamics. This assumption is common in Galerkin approximation control techniques in the literature [4],[6],[7],[8] and holds when the modes of finite dimensional part capture the dominant dynamics of the DPS.

VI. APPLICATION TO DIFFUSION-REACTION PROCESS

In this simulation, the performance of finite-horizon controller is verified on a long, thin rod in a reactor. Under standard assumptions, the spatiotemporal evolution of the dimensionless rod temperature is described by the following parabolic PDE [8] given by

\[
\frac{\partial \bar{x}}{\partial t} = \frac{\partial^{2} \bar{x}}{\partial z^{2}} + \beta_{r} e^{-\gamma/(1+\bar{x})} + \beta_{t} b(z)u(t) - \bar{x} - \beta_{r} e^{-\gamma},
\]

subject to the boundary conditions:

\[
\bar{x}(0,t) = 0, \quad \bar{x}(\pi,t) = 0, \quad \bar{x}(z,0) = \bar{x}_{0}(z),
\]

where \( \bar{x} \), \( \beta_{r} \), \( \gamma \), \( \beta_{t} \), denote dimensionless rod temperature, heat of reaction, activation energy and heat transfer coefficient respectively, \( u(t) \) denotes the vector of inputs and \( b(z) \) the vector of corresponding actuator distribution functions.

The following values are considered as the process parameters: \( \beta_{r} = 50.0, \beta_{t} = 2.0, \gamma = 4.0 \) and a single point control actuator is assumed to be available at the middle of the rod. For these values, it was verified that the operating steady state \( \bar{x}(z,t) = 0 \) is unstable. The control objective therefore is to stabilize the rod temperature profile at the unstable steady state \( \bar{x}(z,t) = 0 \) by manipulating the temperature of the cooling medium which is subject to hard constraints. To achieve this objective, the controlled output is defined as

\[
y_{c}(t) = \int_{0}^{\pi} \sqrt{2/\pi} \sin(z)\bar{x}(z,t)dz.
\]

The eigenvalue problem for the spatial differential operator of the process can be solved analytically and its solution yields

\[
\lambda_{j} = -j^{2}, \quad \phi_{j}(z) = \sqrt{2/\pi} \sin(jz), \ j = 1,\ldots,\infty.
\]

For this system, we consider the first eigenvalue as the dominant one and use standard Galerkin method to derive the finite dimensional part which is used for controller synthesis. In this case, the controlled output (34) renders the necessary state for the identification of system dynamics and controller design as described in parts A and B of Section IV. It should be noted that if more eigenvalues are considered in the finite dimensional part, the necessary states are simply obtained by replacing the kernel in (34) with corresponding eigenfunctions. The controller is then implemented on a 30th order Galerkin discretization of the parabolic PDE system (higher order discretizations led to identical results).

We now proceed with the implementation of the controller. The value function is chosen as \( V(\bar{x}_{1}) = 0.1 + \int_{0}^{\pi} (10\bar{x}_{1}^{2}(t) + \int_{0}^{\bar{u}} \rho^{-1}(v)dv)d\tau \) and the control constraint is chosen to be \( \bar{x}_{1} \). The NN identifier activation function is chosen as \( \sigma_{f} = \rho([1,x_{1},x_{1}^{2},\ldots,x_{1}^{10}]) \) with the bound \( \rho_{max} = 10 \) and for \( \sigma_{c} = [1] \). State dependent part of basis function for approximating

Fig. 1. Evolution of the closed-loop rod temperature under the proposed control scheme.
optimal value function \( V^*(x_1) \) is chosen as \( \rho_1([1,x_1^2,\ldots,x_1^K]) \) with the bound \( \rho_{max}=10 \).

The time dependent part of basis function for approximating optimal value function is designed as \( \rho_2([1,0.1t+9]) \) with the bound \( \rho_{max}=100 \) and it is assumed that \( \phi = \rho_1 \otimes \rho_2 \). The simulation is run with sampling time of 0.004. The tuning parameters are selected as \( \alpha_1 = 0.125, \alpha_2 = 6, \alpha_1 = 25 \) and \( K = 100 \). Input constraint bound is selected as \( \rho_{max}=10 \). Initial conditions are chosen as \( x(0) = [0.3,0.25,0.4,0,-0.11,0\ldots], W_1(0) = [2,\ldots,2] \) and \( W_2(0) = [1,1,1,0\ldots] \).

Fig. 1 shows the evolution of the closed-loop rod temperature. Clearly, the controller successfully stabilizes the temperature profile at the desired steady-state. The results confirm the effectiveness of the controller in regulating performance and control input reduction on a diffusion reaction process as a versatile example of DPS.

**REFERENCES**


