Computational Aspects of a Nonlinear Problem Involving Electromagnetic Transients in Ferromagnetic Shields

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Abstract—Previously, an analytical procedure was developed to characterize the nonlinear electric field transients induced at the inner surface of long, thin-walled, cylindrical, electrically conductive, ferromagnetic shields by short duration, surface current pulses directed axially along the outer surface. The analytical procedure uses mathematical analysis supplemented with numerical calculations. Previous papers emphasized the mathematical aspects of the problem. This paper considers the computational aspects.

Some practical aspects associated with the implementation of a finite difference time-domain (FDTD) formulation are discussed. The effects of spatial and time increments on numerical results are investigated. A method for estimating the residual error in benchmark calculations is proposed and demonstrated, and results of example calculations are presented.

INTRODUCTION

In some applications, ferromagnetic materials offer potential advantages due to their high magnetic permeabilities; however, such materials exhibit nonlinear behavior (including saturation) under intense pulsed electromagnetic field conditions. The field-dependent relative differential permeability is defined as

$$\mu_r(H) = \frac{1}{\mu_0} \frac{dB(H)}{dH},$$

where $H$ is the magnetic field strength, $B(H)$ is the magnetization curve, and $\mu_0=4\pi \times 10^{-7}$ T/m is the permeability of free space. Hysteresis is neglected, and $\mu_r(H)$ is assumed to be a reversible, single-valued function.

The analysis of pulsed electromagnetic field interactions with ferromagnetic shields is complicated by the variation of the differential magnetic permeability with $H$, which makes the phenomenon nonlinear. Previously, an analytical procedure was reported that can be used to calculate the effective permeabilities that correspond to the prescribed $\mu_r(H)$. This paper emphasizes some computational aspects that must be considered in order to achieve meaningful results.

The effects of the spatial increment, $Ax$, and time increment, $At$, on the effective permeabilities are investigated. Empirically, it is found that the error is proportional to $Ax^2$ and to $At$. Based on this observation, an empirical method for estimating the error due to $Ax$ and to $At$ is proposed and demonstrated with selected effective permeability calculations. The method can be used to estimate the residual error in benchmark calculations.

MATHEMATICAL FORMULATION

A planar approximation to the cylindrical problem considers a sheet of thickness, $d$, and electrical conductivity, $\sigma$. The problem is described mathematically by the nonlinear partial differential equation

$$\frac{\partial^2 H}{\partial x^2} = \sigma \mu_r(\frac{\partial H}{\partial t}),$$

where $x$ is the spatial variable and $t$ is the temporal variable. The equation is to be solved subject to three boundary conditions: the initial condition and two boundary conditions. The initial condition is

$$H(x,0) = 0.$$

At the outer surface ($x = 0$), there is an applied field

$$H(0,t) = A \left[ \exp \left( -\frac{t}{\tau_f} \right) - \exp \left( -\frac{t}{\tau_r} \right) \right],$$

where $A$ is the amplitude factor (amperes/meter), $\tau_f$ is a time constant (seconds) that is associated with the rise-time of the applied pulse, and $\tau_r$ is a time constant (seconds) that is associated with the falltime of the applied pulse. At the inner surface ($x = d$), there is a vanishing field condition

$$H(d,t) = 0.$$

Although the magnetic field intensity vanishes at the inner surface, there is an electric field transient at the inner surface. The electric field response at the inner surface ($x = d$) is determined from the spatial partial derivative of $H$ evaluated at the inner surface

$$E(d,t) = -\frac{1}{\sigma} \frac{\partial H(x,d,t)}{\partial x}.$$

It is this transient electric field response that is of interest. The peak value, $E_{peak}$, and the time, $t_{peak}$, at which the peak value occurs are of particular interest.

NUMERICAL FORMULATION

In general, the nonlinear problem posed by (2)-(6) requires numerical analysis. An implicit finite difference formulation using backward time differencing has been used in other studies to model related nonlinear problems.[3][4] In this study, an implicit finite difference formulation using backward time differencing is used to compute the magnetic field intensity, $H(x,t)$. At each time step, the electric field, at the inner surface, $E(d,t)$, is determined by numerical evaluation of the spatial partial derivative of $H(x,t)$ at the inner surface.

In the finite difference formulation of the nonlinear problem, the value of $\mu_r(H)$ depends on which value is ascribed to $H$. In the numerical formulation used in [3] and [4], the argument for the differential permeability is an average of the value of $H$ at the old time step ($t$) and the value of $H$ at the new time step ($t'$)

$$\mu_r = \frac{H(x,t') + H(x,t)}{2}.$$

Using central spatial differencing and backward time
The auxiliary conditions associated with the nonlinear partial differential equation in the present problem can be formulated in a straightforward manner. The initial condition given by (3) becomes
\[ H(x, 0) = 0. \]  
(9)
The applied pulse condition at the outer surface \((x = d)\) given by (4) becomes
\[ \mathcal{H}(0, t_j) = \mathcal{A}_f \left[ \exp \left( -\frac{t_j}{\tau_f} \right) - \exp \left( -\frac{t_j}{\tau_f} \right) \right]. \]  
(10)
The vanishing field condition at the inner surface \((x = 0)\) given by (5) becomes
\[ \mathcal{H}(d, t_j) = 0. \]  
(11)
The implicit formulation yields a relationship that involves quantities at the old time step \((t_0)\), which are known, and quantities at the new time step \((t_1)\), which are to be determined. An iterative approach is used to solve the system of equations.

Once the \(H\) field distribution is determined at a particular time, the electric field at the inner surface is determined numerically by a two-point approximation to the spatial derivative
\[ E(d, t_j) = -\frac{1}{\mathcal{A}} \left( \frac{H(x, t_j) - H(x_1, t_j)}{\Delta x} \right) \mathcal{A}_s. \]  
(12)

IMPLEMENTATION OF THE FDTD FORMULATION

In principle, the above FDTD approach can be applied in a straightforward manner to a general pulse given by (4). For the intense, short-duration pulses considered here, there are some practical aspects that must be considered in the implementation of the FDTD formulation. The shield is divided into \(N_1\) spatial intervals so that
\[ \Delta x = \frac{d}{N_1}. \]  
(13)
There are \(N_1 + 1\) gridpoints \(x_i\) where
\[ \begin{align*}
  x_1 &= 0, \\
  x_2 &= \Delta x, \\
  \vdots \\
  x_{N_1} &= d - \Delta x, \\
  x_{N_1+1} &= d.
\end{align*} \]  
(14)

For the short duration pulses considered here, it is assumed that the applied pulse occurs well before the peak electric field response \(E_{\text{peak}}\) at the inner surface. The finite difference formulation allows the incorporation of different time increments during the course of the analysis. In the present implementation of the finite difference formulation, the numerical analysis is divided into three time ranges, which allows three time increments \((\Delta t_1, \Delta t_2, \text{ and } \Delta t_3)\) to be used in the numerical analysis. Usually,
\[ \Delta t_1 < \Delta t_2 < \Delta t_3. \]  
(15)

It should be noted that \(E_{\text{peak}}\) is delayed due to the time it takes for the effects of the applied pulse to diffuse through the shield. For significant delays, \(\Delta t_2\) may have to be several orders of magnitude larger than \(\Delta t_1\) and \(\Delta t_3\) in order to solve the problem in a reasonable time. Typically, smaller time increments are required for rapidly changing applied pulses. Frequently, the computations are more difficult for large amplitudes and may require the use of smaller time increments. In order to obtain accurate results, the applied pulse must be represented accurately. In particular, it is important that the peak value of the applied pulse be included. The applied pulse reaches a peak value at
\[ t_p = \frac{\ln \left( \frac{t_f}{t_f} \right)}{\ln \left( \frac{1 - \frac{t_f}{\tau_f}}{\frac{t_f}{\tau_f}} \right)}. \]  
(16)
The first time period, \(T_1\) extends from \(t = 0\) to \(t = \tau_{\text{peak}}\) to ensure that the peak value of the applied pulse is included in the numerical analysis. The time period is divided into \(N_1\) time increments so that
\[ \Delta t_1 = \frac{\tau_{\text{peak}}}{N_1}. \]  
(17)
The second time period, \(T_2\) extends from \(t = \tau_{\text{peak}}\) to an arbitrary time \(t = \tau_{\text{mor}}\), which is selected as the end of the applied pulse. At \(t = \tau_{\text{mor}}\), the applied pulse should have decayed to a negligible value in order to mitigate some of the computational difficulties associated with the abrupt change in the boundary condition that can occur if the applied pulse has an appreciable value when the time step is increased. The second time period is divided into \(N_2\) time increments so that
\[ \Delta t_2 = \frac{\tau_{\text{mor}} - \tau_{\text{peak}}}{N_2}. \]  
(18)

In this study, \(t_{\text{mor}}\) was selected to be
\[ t_{\text{mor}} = 100 \tau_f \]  
(19)
by which time even the largest pulses have decayed to a negligible value.

The third time period, \(T_3\) extends from \(t = t_{\text{mor}}\) to the end of the time period of interest \(t = t_{\text{stop}}\), which in the present case is usually shortly after \(E_{\text{peak}}\). The third time period is divided into \(N_3\) time increments so that
\[ \Delta t_3 = \frac{t_{\text{stop}} - t_{\text{mor}}}{N_3}. \]  
(20)

Due to the vanishing field condition at the inner surface, the expression for the electric field reduces to
\[ E(d, t_j) = -\frac{1}{\mathcal{A}} \left( \frac{H(x, t_j) - H(x_1, t_j)}{\Delta x} \right) \mathcal{A}_s. \]  
(21)
A higher order (multipoint) approximation for the first derivative can be used; however, in the present problem it was found that this can lead to numerical artifacts such as reverse polarity excursions in the numerical results. The overall problem involves numerical differentiation of numerical results, which is tenuous at best. Sometimes it is necessary to use very precise calculations involving small \(\Delta x\) and \(\Delta t\) to achieve the desired results.

Since an iterative approach is used, some criterion is needed to determine when to stop the iteration or to reduce the size of the time increment, if necessary. In this study, a criterion based on the sum of the relative differences between results at each iteration, \(k\), and those at the previous iteration, \(k - 1\), is used
\[ \kappa = \sum_{i=1}^{N_1} \frac{|H_i(x, t_j) - H_{i-1}(x, t_j)|}{|H_i(x, t_j)|}. \]  
(22)
The iterations are terminated when
\[ k < \kappa_{\text{max}}. \]  
(23)
The time increment is reduced by a factor of 2 if (23) is not achieved within a prescribed number of iterations. Care must be taken not to make \( \kappa_{\text{max}} \) too small relative to the numerical precision associated with the representation of real numbers in the computational system being used.

**IMPULSE RESPONSE**

A short duration pulse is one for which
\[ t_0 < t_p < t_{\text{peak}}, \]  
(24)
where \( t_{\text{peak}} \) is the time at which the peak value of the electric field response occurs. The effect of such an applied pulse is that of an impulse with magnitude \( Q_o \).

\[ H(0,t) = Q_o \delta(t), \]  
(25)

where \( Q_o \) is the charge per unit circumference that is transported axially along the cylinder during the pulse. For pulses such as those in (4), \( Q_o \), \( (\text{C/m}) \) is given by
\[ Q_o = A(t_f - t_i). \]  
(26)

For an impulse surface current pulse, it was shown in [1] that the peak value and the time of the peak value could be expressed in terms of effective permeabilities that depend on the applied pulse parameter \( \beta \).

\[ \beta = \frac{Q_o}{\sigma d^2}, \]  
(27)

which is a fundamental combination of all of the nonmagnetic problem parameters. An effective permeability is defined as the value of a constant relative permeability that would yield the same result for \( E_{\text{peak}} \) and \( t_{\text{peak}} \).

The effective permeability for the peak value of the electric field, \( \mu_f(\beta) \), is defined as
\[ \mu_f(\beta) = 5.922053727 \frac{Q_o}{\mu_o E_{\text{peak}} \sigma d^2}. \]  
(28)
The effective permeability, \( \mu_f(\beta) \), is the value of the constant relative permeability that would yield the same value for \( E_{\text{peak}} \) from the linear solution for a constant permeability. The effective permeability for the time at which the peak value occurs, \( \mu_f(\beta) \), is defined as
\[ \mu_f(\beta) = 10.899379 \frac{t_{\text{peak}}}{\mu_o \sigma d^2}. \]  
(29)
The effective permeability, \( \mu_f(\beta) \), is the value of the constant relative permeability that would yield the same result for \( t_{\text{peak}} \) from the linear solution for a constant permeability.

**EMPIRICAL ERROR CONSIDERATIONS**

In general, the effective permeabilities \( \mu_f(\beta) \) and \( \mu_f(\beta) \) for a given value of \( \beta \) have to be evaluated numerically, and inevitably there will be some error associated with the approximation. For the FDTD formulation described above, it has been found empirically that the error is proportional to \( (\Delta x)^2 \) and to \( \Delta t \). (Such behavior is typical of a parabolic partial differential equation.) For a given value \( \beta = \beta_o \) the calculated value of \( \mu_f(\beta) \) contains error contributions due to \( \Delta x \) and the three time increments, \( \Delta t_1, \Delta t_2, \) and \( \Delta t_3 \). For small \( \Delta x, \Delta t_1, \Delta t_2, \) and \( \Delta t_3 \), the calculated value for \( \mu_f(\beta) \) (denoted by \( <\mu_f(\beta)> \)) deviates from the actual value as follows

\[ <\mu_f(\beta)> = \mu_f(\beta) + \epsilon_6, \]  
(30)

where the error \( \epsilon_6 \) is approximately
\[ \epsilon_6(\beta) = a_0 (\Delta x)^2 + a_1 \Delta t_1 + a_2 \Delta t_2 + a_3 \Delta t_3 + \ldots \]  
(31)
The coefficients \( a_0, a_1, a_2, \) and \( a_3 \) need to be determined for the applied pulse of interest (\( \beta = \beta_o \)). Similarly,

\[ <\mu_f(\beta)> = \mu_f(\beta) + \epsilon_7, \]  
(32)

where the error \( \epsilon_7 \) is approximately
\[ \epsilon_7(\beta) = b_0 (\Delta x)^2 + b_1 \Delta t_1 + b_2 \Delta t_2 + b_3 \Delta t_3 + \ldots \]  
(33)
The coefficients \( b_0, b_1, b_2, \) and \( b_3 \) need to be determined for \( \beta = \beta_o \) of interest. The error also includes higher order terms (denoted by \( \ldots \)) ; however, those terms shown are observed to be the dominant ones for small \( \Delta x, \Delta t_1, \Delta t_2, \) and \( \Delta t_3 \) The error expression for \( \epsilon_7 \) can be rewritten as
\[ \epsilon_7 = \epsilon_{70} + \epsilon_{71} \Delta t_1 + \epsilon_{72} \Delta t_2 + \epsilon_{73} \Delta t_3 + \ldots \]  
(34)
where \( \epsilon_{70}, \epsilon_{71}, \epsilon_{72}, \) and \( \epsilon_{73} \) are the terms in (31) and \( \epsilon_7 \) as
\[ \epsilon_7 = \epsilon_{70} + \epsilon_{71} \Delta t_1 + \epsilon_{72} \Delta t_2 + \epsilon_{73} \Delta t_3 + \ldots \]  
(35)

**EMPIRICAL ESTIMATION OF THE COEFFICIENTS**

The coefficients \( a_0, a_1, a_2, \) and \( a_3 \) can be estimated by performing a series of calculations in which \( \Delta x, \Delta t_1, \Delta t_2, \) and \( \Delta t_3 \) are varied individually while keeping the others constant and observing the change in the numerical result. The relative change for the \( n \)th case is determined using the operation
\[ \frac{\delta \mu}{\delta (\Delta x)^2} = \frac{\mu_n - \mu_{n-1}}{\Delta x_{n-1}^2 - \Delta x_{n-1}^2}. \]  
(36)

for \( \Delta x \), or the operation
\[ \frac{\delta \mu}{\delta (\Delta t_n)_{j}^2} = \frac{\mu_n - \mu_{n-1}}{(\Delta t_j_{n-1})^2 - (\Delta t_j_{n-1})^2}. \]  
(37)

for \( \Delta t \). The coefficients, \( a_n \), can be approximated as follows

\[ a_0 = \frac{\delta \mu_f(\beta)}{\delta (\Delta x)^2} j \beta_0. \]  
(38)

\[ a_1 = \frac{\delta \mu_f(\beta)}{\delta (\Delta t_1)} j \beta_0. \]  
(39)

\[ a_2 = \frac{\delta \mu_f(\beta)}{\delta (\Delta t_2)} j \beta_0. \]  
(40)

\[ a_3 = \frac{\delta \mu_f(\beta)}{\delta (\Delta t_3)} j \beta_0. \]  
(41)

In practice, a series of calculations are performed in which \( \Delta x, \Delta t_1, \Delta t_2, \) and \( \Delta t_3 \) are progressively reduced until the results for \( a_0, a_1, a_2, \) and \( a_3 \) are reasonably constant. A similar approach can be used with \( \mu_f(\beta) \) to estimate the coefficients \( b_0, b_1, b_2, \) and \( b_3 \).

**ERROR ESTIMATION**

Results from a series of calculations for progressively smaller \( \Delta x \) and \( \Delta t_1, \Delta t_2, \) and \( \Delta t_3 \) can be used to estimate the residual error and to adjust the calculated results for \( \mu_f(\beta) \) and \( \mu_f(\beta) \). From (30), the calculated value for \( \mu_f(\beta) \) can be adjusted for the effects of \( \Delta x, \Delta t_1, \Delta t_2, \) and \( \Delta t_3 \) as follows
\[ \mu_f(\beta) = <\mu_f(\beta)> - \epsilon_6, \]  
(32)

and from (32)
\[ \mu_f(\beta) = <\mu_f(\beta)> - \epsilon_7. \]  
(33)

This is useful for estimating the residual error in benchmark calculations. Inspection of the individual terms in \( \epsilon_6 \) and \( \epsilon_7 \) is useful in deciding how to allocate computational resources so the the errors associated with \( \Delta x, \Delta t_1, \Delta t_2, \) and \( \Delta t_3 \) are comparable.
Example

For nonlinear problems, there are few analytical results that can be used for comparison to estimate the error in numerical calculations; however, for pulses with a sufficiently small amplitude, the differential permeability does not depart appreciably from its initial value. In this case, the numerical results can be compared with analytical results for a constant permeability.

To illustrate some of the computational aspects discussed above, a simple $\mu_d(H)$ is considered. A simple magnetization curve is

$$B(H) = B_s \left[ 1 - \exp \left( -\frac{H}{H_s} \right) \right], \quad (44)$$

where $B_s$ is the saturation flux density and $H_s$ is a characteristic value. From (1), the corresponding relative differential permeability is

$$\mu_d(H) = \mu_r \exp \left( -\frac{H}{H_s} \right), \quad (45)$$

where the initial value $\mu_r$ is defined as

$$\mu_r = \frac{B_s}{H_s}. \quad (46)$$

To demonstrate the method described above, the initial value of $\mu_d(0)$ and $\mu_d(H)$ were calculated for the problem parameters shown in Table I, which were chosen for purposes of illustration rather than physical realism.

Since the amplitude of the applied pulse

$$A_x = 1 \times 10^{-10} \text{A/m} \quad (47)$$

is exceedingly small, $\mu_d(H)$ never undergoes a significant departure from $\mu_d(0)$; consequently, the results should be near those for a constant permeability of one. The corresponding applied pulse parameter $\beta$ is

$$\beta = \frac{Q}{\sigma A} = \frac{A(t_f - t_i)}{\sigma A} = 9 \times 10^{-12} \quad (48)$$

which is quite small; consequently, it is expected that the values for $\mu_d(0)$ and $\mu_d(H)$ should be unity.

A series of FDTD calculations were performed using the standard computational conditions shown in Table II for each case. The error analysis technique is illustrated for $\mu_d(H)$; however, a similar approach can be used for $\mu_d(0)$. In the series of calculations, $\Delta x$, $\Delta t_i$, $\Delta t_p$, and $\Delta t_f$ were progressively reduced individually (with the others fixed) until the results for $a_x$, $a_t$, $a_s$, and $a_r$ were reasonably constant. The operation for $\Delta x$ is illustrated in Table III from which

$$a_x = 2.72470. \quad (49)$$

The operation for $\Delta t_i$ is shown in Table IV from which

$$a_t = 3.87108 \times 10^{10}. \quad (50)$$

The operation for $\Delta t_p$ is shown in Table V from which

$$a_s = 3.87150 \times 10^{10}. \quad (51)$$

The operation for $\Delta t_f$ is shown in Table VI from which

$$a_r = 1.28981 \times 10^{10}. \quad (52)$$

The most accurate calculations presented are those for $\Delta x=1 \times 10^{-3}$, $\Delta t_i=2.55843 \times 10^{-11}$, $\Delta t_p=9.97442 \times 10^{-11}$, and $\Delta t_f=1 \times 10^{-10}$ for which the calculated value is

$$\mu_d(0) = 1.000005684247. \quad (53)$$

Compared to $\mu_d(0)\times 1$, the actual numerical error is

$$\epsilon_{err} = \mu_d(\beta) - \mu_d(0) = 5.684247 \times 10^{-4}. \quad (54)$$

The individual error contributions are estimated to be

$$\epsilon_{e_x} = a_x(\Delta x)^2 = (2.72470)(1 \times 10^{-3})^2 = 2.72470 \times 10^{-6}. \quad (55)$$

$$\epsilon_{e_t} = a_t(\Delta t_i)^2 = (-3.87108 \times 10^{-10})(2.55843 \times 10^{-7}) = -9.90389 \times 10^{-7}. \quad (56)$$

$$\epsilon_{e_p} = a_p(\Delta t_p)^2 = (3.87150 \times 10^{-10})(9.97442 \times 10^{-11}) = 3.86160 \times 10^{-6}. \quad (57)$$

$$\epsilon_{e_f} = a_f(\Delta t_f)^2 = (1.28981 \times 10^{10})(1 \times 10^{-14}) = 1.28981 \times 10^{-7}. \quad (58)$$

The total estimated error is

$$\epsilon = 2.72470 \times 10^{-6} + 9.90389 \times 10^{-7} + 3.86160 \times 10^{-6} + 1.28981 \times 10^{-7} + \ldots = 5.72469 \times 10^{-6}. \quad (59)$$

which is in good agreement with the actual error (54).

From the above analysis, one can gain some insight into the sources of error and the allocation of computational effort to further reduce the error. For example, most of the residual error in $\mu_d(\beta)$ can be attributed to $\epsilon_{e_x}$ for $\Delta x$ and to $\epsilon_{e_t}$ for $\Delta t_i$. The contribution due to $\epsilon_{e_p}$ is relatively small; consequently, further reduction in $\Delta t_f$ alone would not significantly reduce the total error. (Conversely, it should be noted that the accuracy of $\mu_d(H)$ is largely determined by $\Delta t_f$; consequently, $\Delta t_i$ and $\Delta t_f$ do not affect $\mu_d(H)$ significantly.)

Conclusion

The finite difference time domain (FDTD) technique can be used for very precise computations for a nonlinear problem involving electromagnetic transients in ferromagnetic shields. To a good approximation, the error in the calculations for effective permeabilities $\mu_d(0)$ and $\mu_d(H)$ was found to be proportional to $(\Delta x)^2$ and to $\Delta t$ for small $\Delta x$ and $\Delta t$. An empirical method for estimating the residual error in $\mu_d(0)$ and $\mu_d(H)$ due to small $\Delta x$ and $\Delta t$ was proposed and demonstrated, and it was found to be in good agreement with the actual error for selected calculations.

References


**Table I**

**STANDARD PROBLEM PARAMETERS FOR EXAMPLE CALCULATIONS**

<table>
<thead>
<tr>
<th>Shield Parameters</th>
<th>Applied Pulse</th>
<th>Applied Pulse Parameter</th>
</tr>
</thead>
<tbody>
<tr>
<td></td>
<td>$H(0, t) = A_s \left[ \exp \left( -\frac{t}{t_f} \right) - \exp \left( -\frac{t}{t_r} \right) \right]$</td>
<td></td>
</tr>
<tr>
<td>Relative Differential Permeability</td>
<td>$\mu_{rd}(H) = \mu_{ri} \exp \left( -\frac{H}{H_1} \right)$</td>
<td></td>
</tr>
<tr>
<td>Electrical Conductivity</td>
<td>$\sigma$ (S/m)</td>
<td></td>
</tr>
<tr>
<td>Wall Thickness</td>
<td>$d$ (m)</td>
<td></td>
</tr>
<tr>
<td>Amplitude</td>
<td>$A_s$ (A/m)</td>
<td></td>
</tr>
<tr>
<td>Rise time</td>
<td>$t_r$ (s)</td>
<td></td>
</tr>
<tr>
<td>Fall time</td>
<td>$t_f$ (s)</td>
<td></td>
</tr>
<tr>
<td>$\beta = \frac{A_s (t_r - t_f)}{\sigma d^2}$</td>
<td></td>
<td></td>
</tr>
</tbody>
</table>

<table>
<thead>
<tr>
<th>$B_s$ (T)</th>
<th>$H_s$ (A/m)</th>
<th>$\mu_{ri} = \frac{B_s}{\mu_0 H_1}$</th>
<th>$\sigma$ (S/m)</th>
<th>$d$ (m)</th>
<th>$A_s$ (A/m)</th>
<th>$t_r$ (s)</th>
<th>$t_f$ (s)</th>
<th>$\beta$</th>
</tr>
</thead>
<tbody>
<tr>
<td>1</td>
<td>$\frac{1}{4\pi \times 10^{-7}}$</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1</td>
<td>1x10^{-10}</td>
<td>1x10^{-12}</td>
<td>1x10^{-11}</td>
</tr>
</tbody>
</table>

**Table II**

**STANDARD COMPUTATIONAL CONDITIONS FOR EXAMPLE CALCULATIONS**

<table>
<thead>
<tr>
<th>Time of Peak Value</th>
<th>First Time Period</th>
<th>End of Applied Pulse</th>
<th>Second Time Period</th>
<th>Stopping Time</th>
<th>Third Time Period</th>
<th>Iteration Criterion</th>
</tr>
</thead>
<tbody>
<tr>
<td>$t_p$</td>
<td>$T_1 = t_p - 0$</td>
<td>$t_{end} = 100 t$</td>
<td>$T_2 = t_{end} - t_p$</td>
<td>$t_{stop}$</td>
<td>$T_3 = t_{stop} - t_{end}$</td>
<td>$\kappa_{max}$</td>
</tr>
<tr>
<td></td>
<td>2.55843x10^{-12}</td>
<td>2.55845x10^{-12}</td>
<td>1x10^{-9}</td>
<td>9.97442x10^{-10}</td>
<td>1.51x10^{-7}</td>
<td>1.50x10^{-7}</td>
</tr>
</tbody>
</table>

**Table III**

**EFFECT OF $\Delta x$ ON THE CALCULATED VALUES OF $\mu_x$ AND $\mu_T$ FOR AN EXPONENTIAL PERMEABILITY $\mu_{rd}(H) = \mu_{rd}(H/H_1)$**

<table>
<thead>
<tr>
<th>Effect of $\Delta x$</th>
<th>Standard Problem Parameters: See Table I</th>
<th>Standard Computational Conditions: See Table II</th>
<th>Fixed Computational Parameters: $N_i = 1 \times 10^5$, $\Delta t = 2.55845 \times 10^{-17} \text{s}$; $N_i = 1 \times 10^5$, $\Delta t = 9.97442 \times 10^{-17} \text{s}$; $N_i = 1.5 \times 10^5$, $\Delta t = 1 \times 10^{-17} \text{s}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$N_0$</td>
<td>$\Delta x$</td>
<td>$(\Delta x)^2$</td>
<td>$\mu_x$</td>
</tr>
<tr>
<td>5</td>
<td>2x10^{-1}</td>
<td>4.0x10^{-2}</td>
<td>1.088828296038</td>
</tr>
<tr>
<td>10</td>
<td>1x10^{-1}</td>
<td>1.0x10^{-2}</td>
<td>1.026157316171</td>
</tr>
<tr>
<td>20</td>
<td>5x10^{-2}</td>
<td>2.5x10^{-3}</td>
<td>1.006750594778</td>
</tr>
<tr>
<td>50</td>
<td>2x10^{-2}</td>
<td>4.0x10^{-4}</td>
<td>1.001091246114</td>
</tr>
<tr>
<td>100</td>
<td>1x10^{-2}</td>
<td>1.0x10^{-4}</td>
<td>1.000275335965</td>
</tr>
<tr>
<td>200</td>
<td>5x10^{-3}</td>
<td>2.5x10^{-5}</td>
<td>1.000071071981</td>
</tr>
<tr>
<td>500</td>
<td>2x10^{-3}</td>
<td>4.0x10^{-6}</td>
<td>1.000013858347</td>
</tr>
<tr>
<td>1000</td>
<td>1x10^{-3}</td>
<td>1.0x10^{-6}</td>
<td>1.000005684247</td>
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</tbody>
</table>

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### Table IV
**Effect of $\Delta t_1$ on the Calculated Values of $\mu_3$ and $\mu_4$ for an Exponential Permeability $\mu_{sd}(H) = \mu_s \exp(-H/H_s)$**

<table>
<thead>
<tr>
<th>$N_1$</th>
<th>$\Delta t_1$</th>
<th>$\mu_3$</th>
<th>$\delta = \frac{\delta \mu_3(\beta)}{\delta[(\Delta t_1)]}$</th>
<th>$\mu_4$</th>
<th>$\delta = \frac{\delta \mu_4(\beta)}{\delta[(\Delta t_1)]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.0\times 10^3$</td>
<td>$2.55843 \times 10^{-16}$</td>
<td>0.999907693113</td>
<td>1.0000094402535</td>
<td>1.000094402535</td>
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</tr>
<tr>
<td>$1.0\times 10^4$</td>
<td>$2.55843 \times 10^{-16}$</td>
<td>0.999996770750</td>
<td>$-3.86859 \times 10^{-10}$</td>
<td>1.000094402535</td>
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</tr>
<tr>
<td>$1.0\times 10^5$</td>
<td>$2.55843 \times 10^{-17}$</td>
<td>1.000005684247</td>
<td>$-3.87108 \times 10^{-10}$</td>
<td>1.000094402535</td>
<td></td>
</tr>
</tbody>
</table>

### Table V
**Effect of $\Delta t_2$ on the Calculated Values of $\mu_3$ and $\mu_4$ for an Exponential Permeability $\mu_{sd}(H) = \mu_s \exp(-H/H_s)$**

<table>
<thead>
<tr>
<th>$N_2$</th>
<th>$\Delta t_2$</th>
<th>$\mu_3$</th>
<th>$\delta = \frac{\delta \mu_3(\beta)}{\delta[(\Delta t_2)]}$</th>
<th>$\mu_4$</th>
<th>$\delta = \frac{\delta \mu_4(\beta)}{\delta[(\Delta t_2)]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.0\times 10^6$</td>
<td>$9.97442 \times 10^{-16}$</td>
<td>1.000040438577</td>
<td>1.000094402535</td>
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<tr>
<td>$1.0\times 10^7$</td>
<td>$9.97442 \times 10^{-17}$</td>
<td>1.000005684247</td>
<td>$3.87150 \times 10^{-10}$</td>
<td>1.000094402535</td>
<td></td>
</tr>
</tbody>
</table>

### Table VI
**Effect of $\Delta t_3$ on the Calculated Values of $\mu_3$ and $\mu_4$ for an Exponential Permeability $\mu_{sd}(H) = \mu_s \exp(-H/H_s)$**

<table>
<thead>
<tr>
<th>$N_3$</th>
<th>$\Delta t_3$</th>
<th>$\mu_3$</th>
<th>$\delta = \frac{\delta \mu_3(\beta)}{\delta[(\Delta t_3)]}$</th>
<th>$\mu_4$</th>
<th>$\delta = \frac{\delta \mu_4(\beta)}{\delta[(\Delta t_3)]}$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$1.5\times 10^6$</td>
<td>$1.0 \times 10^{-13}$</td>
<td>1.00006845073</td>
<td>1.000095356580</td>
<td></td>
<td></td>
</tr>
<tr>
<td>$1.5\times 10^7$</td>
<td>$1.0 \times 10^{-14}$</td>
<td>1.00006845073</td>
<td>$1.28981 \times 10^7$</td>
<td>1.000094402535</td>
<td>$1.06005 \times 10^7$</td>
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</tbody>
</table>