An analytic Fournier-Forand scattering phase function as an alternative to the Heney-Greenstein phase function in hydrologic optics

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Abstract — This work presents results of numerical analysis of the Fournier-Forand scattering phase function as an alternative to the Heney-Greenstein phase function in hydrologic optics. A number of equations are derived that connect different integral parameters of the Fournier-Forand phase function, including the normalization coefficient, with the parameters of this function. The Mathematica and FORTRAN programs, that computes the Fournier-Forand scattering phase function and some of its integral characteristics, are released for public use.

INTRODUCTION

The Heney-Greenstein (HG) phase function, that was originally proposed for use in astrophysics [1], has become very popular in hydrologic optics, including some practically important underwater target detection algorithms. The HG phase function is appealing because it is very simple and convenient for mathematical analysis.

Other analytical phase functions that have been proposed for radiative transfer calculations are: 1) A three-parameter analytic phase function, a combination of forward and backward elongated HG phase functions proposed by G. W. Kattawar [2]. 2) A Haltrin phase function, a combination of hyperbolic and delta- functions [3], that gives an analytic asymptotic solution to the radiative transfer equation in the form of Heney-Greenstein function with the elongation parameter depending on the inherent optical properties of the medium. 3) A Cornette-Shanks phase function [4] that converges to the Rayleigh phase function when the average cosine (μ) << 1 and approaches the HG phase function when 1 - μ << 1. 4) A Reynolds-McCormick two parameter phase function that generalizes the Heney-Greenstein phase function [5] to a hyperbolic one.

When used in ocean optics all listed phase functions with all their advantages have one major shortcoming: their shapes don’t resemble at all the shapes of realistic marine phase functions [6].

The two-parametric analytic Fournier-Forand (FF) scattering phase function was proposed in 1994 to the ocean optics community [7]. It was almost unnoticed by the optical oceanographic community possibly because the FF phase function has a more complex analytic form and the original paper did not include analysis of its properties. The major advantages of the Fournier-Forand phase function are: 1) it depends only on two parameters; and 2) it approximates almost all realistic marine phase functions with a very high degree of precision.

ORIGINAL FORMULA

The one-parametric analytic Fournier-Forand scattering phase function has the following form [7, 8]:

\[ p(\mu) = \frac{A(1 + \mu^2)}{(1 - \delta^2)^{3/2}} \left[ (1 - \delta) - 1 + \delta \right]^+ \]

\[ = \frac{2}{(1 - \mu)^{3/2}} \left[ 1 - \delta^{-0} - (1 + w)(1 - \delta) \right], \]

where \( A \) is a normalization factor, and

\[ w = \frac{2 - \nu}{2}, \quad \delta = \frac{2(1 - \mu)}{3(n - 1)}, \quad 3.5 \leq \nu \leq 5, \]

and \( \mu = \cos \theta \), \( \theta \) is a scattering angle. The two parameters of this phase function are: 1) \( n \), relative to water refraction index of scattering particles, and 2) \( \nu \), a Junge parameter in the size distribution of particles used for derivation of Eq.(1). The phase function (1) is normalized according to the following rule:

\[ 0.5 \int_0^\pi p(\mu) d\mu = 1. \]

Expressions for several parameters of the FF phase function are given in the next section. They include: the average cosine,

\[ \langle \mu \rangle = 0.5 \int_0^\pi p(\mu) \mu d\mu = 1, \]

the backscattering probability,

\[ B = 0.5 \int_0^\pi p(\mu) d\mu, \]

the average scattering angle,

\[ \langle \theta \rangle = 0.5 \int_0^\pi p(\mu) \cos^{-1}(\mu) d\mu, \]

and the average square of the scattering angle,

\[ \langle \theta^2 \rangle = 0.5 \int_0^\pi p(\mu) \cos^{-1}(\mu)^2 d\mu. \]

INTEGRAL CHARACTERISTICS

The normalization factor \( A \) in Eq. (1), and parameters \( B, \langle \mu \rangle, \langle \theta \rangle \) and \( \langle \theta^2 \rangle \) were calculated using the Mathematica code given in the APPENDIX 1. The normalization constant \( A \) is represented in the form of the following regression:
The calculated regression coefficients $A_j$ in Eq. (8) are given in Table 1.

**Table 1. Coefficients $A_j$**

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<thead>
<tr>
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<th>0</th>
<th>1</th>
<th>2</th>
<th>3</th>
<th>4</th>
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</thead>
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<td>1.0898</td>
<td>-.24209</td>
</tr>
</tbody>
</table>

The average cosine is represented as the following regression:

$$\langle \mu \rangle = 10^3 \sum_{i,j=0}^4 A_j n'(5-v)^{i+j}$$

and the coefficients $A_j$ are given in the following table:

**Table 2. Coefficients $A_j$**

<table>
<thead>
<tr>
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<td>5.2776</td>
<td>-7.2304</td>
<td>4.4071</td>
<td>-0.9918</td>
</tr>
</tbody>
</table>

The backscattering probability is given by the following formula:

$$B = 0.5 - 10^3 \sum_{i,j=0}^4 B_j n'(5-v)^{i+j}$$

and the coefficients $B_j$ are given in the following table:

**Table 3. Coefficients $B_j$**

<table>
<thead>
<tr>
<th>i</th>
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<th>2</th>
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<td>-3.479</td>
<td>0.6599</td>
</tr>
</tbody>
</table>

The average square of the scattering angle is represented as:

$$\langle \theta^2 \rangle = \frac{\pi^2}{2} - 10^3 \sum_{i,j=0}^4 \theta_{ij} n'(5-v)^{i+j}$$

and the coefficients $\theta_{ij}$ are given in the following table:

**Table 4. Coefficients $\theta_{ij}$**

<table>
<thead>
<tr>
<th>i</th>
<th>0</th>
<th>1</th>
<th>2</th>
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</tr>
</tbody>
</table>

**ILLUSTRATIONS**

The angular distributions of the Fournier-Forand scattering function are calculated using the FORTRAN code given in the APPENDIX 2. Several examples of calculated FF phase functions are shown in Figure 1. All calculated examples strikingly resemble experimental phase functions measured by different authors and presented in Ref. [6].

![Figure 1. Examples of normalized Fournier-Forand phase functions for a set of parameters $(v,n)$.](image-url)
CONCLUSION

Our model calculations confirm conclusion of Ref. [7] that in approximating experimental oceanic phase functions the Fournier-Forand phase function is preferable to the phase functions listed in the INTRODUCTION. Consequently it is more suitable for use in radiative transfer models in seawater.

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APPENDIX 1: MATHEMATICA CODE TO CALCULATE INTEGRAL PARAMETERS OF THE FOURNIER-FORAND PHASE FUNCTION

```
n=1.25;
u=4.0;
dmu=a/(3*(n-1)^2);
uf=4*(n-2)/(n-1)^2;
ed[mu]=a[u[mu]]*[u[mu]]^2;
w[mu]=(1-d[mu])^2-d[mu]*(1+u[mu])^2;
p0[mu]=0.5*(1+mu*mu)*[(ed[mu])^2ualethet];
A0=N[Integrate[p0[mu],{mu,-1,1}]]; p[mu]=0.5*A0*(1+mu*mu)[w[mu]];
B=N[Integrate[p[mu],{mu,-1,1}]]; muAv=N[Integrate[mu*p[mu],{mu,-1,1}]]; thAv=N[Integrate[ArcCos(mu)*p[mu],{mu,-1,1}]]; thSqAv=N[Integrate[ArcCos(mu)*ArcCos(mu)*p[mu],{mu,-1,1}]];
```

```
Print["A0=",A0]
Print["B=",B]
Print["<q1>=",muAv]
Print["<q2>=",thAv]
Print["<q2>=",thSqAv]
```

REFERENCES


APPENDIX 2: A FORTRAN CODE TO CALCULATE THE FOURNIER-FORAND PHASE FUNCTION OF SCATTERING

```
! real function ffphf(mu,n,thet,A0)
! Computes a normalized Fournier-Forand phase function of scattering. This code is based on the Eq.(8) of this paper, i.e. on the corrected [5] version of the original equation by G.R.Fournier and J.L.Forand [7]. A0 should be precomputed with the code given above or with the Eq.(8) of this paper (less precise).
! implicit none
real mu,n,thet,A0, v, ev, dn, d2
real u2,v,delta, ed, dv, s1, s2, px

v = 0.5*(3.-nu)
ev = 1.+v
dn = n-1.
d2 = dn*n
mu = COSD(thet)
u2 = 2*(1.-mu)
u = SQRT(u2)
delta = u2/(3.*d2)
ed = 1.-delta
dv = delta*v
s1 = v*ed-1.+dv
s2 = 1.-delta*dv-ev*ed
s2 = 4*s2/u2
px = s1*s2
fphf = A0*px*(1.+mu*mu)/s2
return
end
```

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