Reliability Calculation Model for Repairable Systems Considering Failure Correlation and Variable Hazard rate

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Abstract—Considered the positive correlation existed in working life lengths of components and repair time lengths of the failure units belonged to the same repairable system based on Copula theories, and extended the distributions for working time lengths and repair time lengths of the components to the general continuous types, rather than confined to exponential distribution. Presented the concept of one step state transition matrix for repairable systems in minimal time difference \( t \to t+\Delta t \), then turned out the state transition density matrix, by solving a system state equation, given availability at any time \( t \) and steady-state availability computing models for single-unit, series type, parallel repairable system consisting of two different units and a repairman respectively. A practical case calculation demonstrates the feasibility and effectiveness of the theory.

Keywords-Copula; correlation; repairable systems; one step state transition matrix; availability

I. INTRODUCTION

The reliability analysis and calculation for the repairable system is always one of the active subject of reliability theory research and its application[1~8]. Suppose that \( X_i \) and \( Y_i \) be independent of each other and submit to exponential distribution. \( X_i \) is working life length of the \( i \)th component in a system, With its counterpart, \( Y_i \) is repair time length of the \( i \)th component. In addition, the assumption of probability that there are two or more components fell into failure state in the same instant would be zero are required. Under these conditions, Classical calculation methods and analysis theories are more mature and used widely [9~11].

In practical projects, due to system objects, work environment, human factors, maintenance factors such as different levels, resulting in repairable system complicated. Generalized Markov process, geometric process, method of supplementary variable were used to study reliability indexes of N-Unit Series Repairable System, two-units series system, cold standby repairable system respectively in [1~3]. Reference [4,5] studied two-units parallel repairable system and a one unit repairable system with multiple vacations of one repairman whose components have two types of failure. Richard[6] and Zhang[7] calculated the reliability of a repairable circular k/n(G) system by the generalized transition probability, and Zhang analyzed the reliability and replacement policy of a k/n(F) system with repairable repair equipment by using the geometric process, the vector Markov process and the queuing theory in [8]. The assumptions that working life length \( X_i \) and repair time length \( Y_i \) for are subject to exponential distribution is necessary, which only reveal the state of component changes by chance, however, for mechanical, electrical or structural systems which fall in the failure state caused by wear, corrosion, fatigue factors etc, obviously the hazard rate of their components are not constant, but rather a single increasing function in working time \( t \). Similarly, the correlations existed in \( X_i \) and \( Y_i \) were not taking into account due to modeling difficulties in [1~11]. Actually, On the one hand, because there are external shocks, that the existence of common cause failure, such as accidental unexpected events, environmental change, human disturbance, etc. for a specific example, as a computer network system, the failure events of each computer are not independent resulted from the power supply circuit, transmission of the virus and other reasons. Overloads, working environment have the same impact on the working lives of components which bear the same task in a mechanical system, and lead to the number of failure components may be more than one in a transient moment. On the other hand, the impact from within the system, namely dependent failures often happen, if a rocket engine explosion, it will cause the other engine failure in a rocket [12].

Therefore, it is very important to research the reliability of a failure correlation and varying hazard rate repairable system, it also has good application advantages. It was the first time to introduce Copula theory to the reliability analysis methods for a mechanical system in [13], and resolved the problem of reliability calculation involving failure correlation for the typical not-repairable series or parallel systems. In this paper, based on Copula correlation theories, the concept of one step state transition matrix \( \mathbf{P}(\Delta t) \) for repairable systems in minimal time difference \( t \to t+\Delta t \) was presented, and then got the state transition density matrix, So the availability calculation models at any time \( t \) for several typical repairable systems were given by solving the system state equation. The problems of modeling and calculating the reliability of repairable systems, considered simultaneously the correlation
among in the working life lengths $X_i$ and repairing time lengths $Y_i$ of components, and promoted them to the general continuous distributions.

II. MODEL ASSUMPTION

Let $X_i$, $i = 1, 2, \cdots, n$ be the working life length of the $i$th component in a specific system, its hazard rate $\lambda_i(t)$ is a function in working time $t$. Obviously, under the same working environment, carrying the same task, result in the correlation structures between $X_i$ were the positive correlation Copula $C_X(u_1, u_2, \cdots, u_n)$ due to common cause failure or dependent failure, see Nelesen[13].

**Definition** [13]. A two-dimensional Copula is a distribution function $C([0, 1]^2 \rightarrow [0, 1])$, with the following properties:

i. $C(u, 0) = C(0, v) = 0$, $C(u, 1) = u$, $C(1, v) = v$.  (1)

ii. For $0 \leq u_i \leq 1$, $0 \leq v_i \leq 1$, $C$ is 2-increasing

$$C(u_1, v_1) - C(u_1, v_1) - C(u_1, v_1) + C(u_1, v_1) \geq 0.$$ (2)

**Theorem** [13]. Let $(X_1, X_2, \cdots, X_n)$ be an $n$ dimensional random vector, with margins $F_1(\cdot), F_2(\cdot), \cdots, F_n(\cdot)$. Then there exists only an $n$-Copula $C_{X}(u_1, u_2, \cdots, u_n)$ such that for all $\mathbf{x}$ in $R^n$, such that

$$P(X_1 \leq x_1, X_2 \leq x_2, \cdots, X_n \leq x_n) = C_{X}(F_1(x_1), F_2(x_2), \cdots, F_n(x_n)).$$ (2)

The Copula function $C_{X}(u_1, u_2, \cdots, u_n)$ reflects the associated structures of random variable $X$, $Y$, and connects the marginal distributions of $X$, $Y$ to their joint distribution, where $F_1(\mathbf{x}) = 1 - \exp(-\int \lambda_i(t) dt)$.  

Similarly, due to the common causes of failure, under the same repair equipment services, maintenance workers experience and other factors, the correlation structures between repair time length $Y_i$ ($i = 1, 2, \cdots, m$) were the positive correlation Copula $C_Y(v_1, v_2, \cdots, v_m)$, and let $\mu_i(t)$ be repair rate of $Y_i$. Assume the failure components immediately access to maintenance state, and follow the first failure first repair principle. The following table lists the types of positive correlation Copula (see in Nelsen [14]).

Here, $\theta$ is the correlation parameter in Copula functions, with the change in working conditions, correlation structure $C_\theta$ and the correlation parameters $\theta$ are different.

III. CALCULATION MODEL FOR SINGLE-UNIT REPAIRABLE SYSTEM

Let the state of a single-unit repairable system which composed of one unit and one repairman be

"0" for system is in working state, "1" for system is in fault state. $\lambda(t)$ and $\mu(t)$ are the system hazard rate and repair rate.

<table>
<thead>
<tr>
<th>Model</th>
<th>Range for $\theta$</th>
<th>$C_\theta(u_1, u_2, \cdots, u_n)$</th>
</tr>
</thead>
<tbody>
<tr>
<td>Gumbel</td>
<td>$(0, 1]$</td>
<td>$\exp\left(-\left(\log u_2\right)^\frac{1}{\theta}\right)$ + $\left(-\log u_1\right)^\frac{1}{\theta}$ + $\cdots$ + $\left(-\log u_n\right)^\frac{1}{\theta}$</td>
</tr>
<tr>
<td>Clayton</td>
<td>$(0, \infty)$</td>
<td>$\left(u_1^{-\theta} + u_2^{-\theta} + \cdots + u_n^{-\theta} - 1\right)^{-1/\theta}$</td>
</tr>
<tr>
<td>Frank</td>
<td>$(0, \infty)$</td>
<td>$\frac{1}{\theta}\log\left[1 + \frac{\theta}{\theta - 1}\left(1 - u_1^{-\theta}</td>
</tr>
</tbody>
</table><p>ight) - \frac{1}{\theta - 1}\left(1 - u_2^{-\theta}\right) - \cdots - \frac{1}{\theta - 1}\left(1 - u_n^{-\theta}\right)\right]$ |</p>

We first find the one step state transition matrix $P(\Delta t)$ for repairable systems in minimal time difference $t \rightarrow t + \Delta t$ (the purpose of giving one step state transition matrix is to obtain the state transition density matrix $Q$, the probability that the same unit occurs state change twice or more in minimal time difference $\Delta t$ would be zero).

$$P(\Delta t) = \begin{pmatrix} p_{00}(\Delta t) & p_{01}(\Delta t) \\ p_{10}(\Delta t) & p_{11}(\Delta t) \end{pmatrix} = \begin{pmatrix} e^{-\lambda \int_{t}^{t+\Delta t}} & e^{\theta \int_{t}^{t+\Delta t}} \\ -\theta \int_{t}^{t+\Delta t} & e^{-\lambda \int_{t}^{t+\Delta t}} \end{pmatrix}. \quad (3)$$

The matrix element $p_{ij}(\Delta t)$ represents the probability that the state of system is $i$ at $t$ and change to $j$ until $t + \Delta t$.

Using $q_{ij} = \lim_{\Delta t \rightarrow 0} \frac{1 - p_{ij}(\Delta t)}{\Delta t}$, $q_{ij} = \lim_{\Delta t \rightarrow 0} \frac{p_{ij}(\Delta t)}{\Delta t}$ (See Lin [15]). With L'Hospital rules, and then get the system state transition density matrix $Q$.

$$Q = \begin{pmatrix} q_{00} & q_{01} \\ q_{10} & q_{11} \end{pmatrix} = \begin{pmatrix} -\lambda(t) & \lambda(t) \\ \mu(t) & -\mu(t) \end{pmatrix}. \quad (4)$$

Let the state vector of the system at time $t$ be $\mathbf{p}(t) = [p_0(t), p_1(t)]$, here $p_0(t), p_1(t)$ represent the probability of the system is in state “0” and state “1” respectively. We may set its initial state be $\mathbf{p}(0) = [1, 0]$. According to equation of state

$$\frac{dp(t)}{dt} = \mathbf{p}(t) \cdot Q. \quad (5)$$

By Laplace transform, method of partial-fraction expansion and the inverse Laplace transform, obtain the availability of system at $t$ using the solution of equation (5), namely
\[ A(t) = p_0(t) = \frac{\mu(t)}{\lambda(t) + \mu(t)} + \frac{\lambda(t)}{\lambda(t) + \mu(t)} \exp\left(-\left(\lambda(t) + \mu(t)\right)t\right) \]  
(6)

The steady state availability be \[ A(\infty) = \frac{\mu(\infty)}{\lambda(\infty) + \mu(\infty)}. \]

IV. CALCULATION MODEL FOR N-UNIT SERIES REPAIRABLE SYSTEM

Series system reliability depends on the reliability of the weakest link. Therefore, the optimization design of system reliability should be reliability of each unit is almost the same. We may try to analyze a series repairable system consisting of the \( n \)-identical units and a repairman, similar to \( n \)-different units, but the situation more complicated.

Let state “\( i \)” represents there were \( i \) components in failure state in this system, \( i = 1, 2, \ldots, n \). The correlation structures between \( X_i \) are the positive correlation Copula \( C_X(u_1, u_2, \ldots, u_n) \), the hazard rate of every component is \( \lambda(t) \). The correlation structures between \( Y_i \) are the positive correlation Copula \( C_Y(v_1, v_2, \ldots, v_m) \), \( m \) is the number of failure components, and \( \mu(t) \) is repair rate of repairable component, then one step state transition matrix \( P(\Delta t) \) for repairable systems in minimal time difference \( t \to t + \Delta t \) was

\[
P(\Delta t) = \begin{bmatrix}
p_{00}(\Delta t) & p_{01}(\Delta t) & \cdots & p_{0m}(\Delta t) \\
p_{10}(\Delta t) & p_{11}(\Delta t) & \cdots & p_{1m}(\Delta t) \\
\vdots & \vdots & \ddots & \vdots \\
p_{n0}(\Delta t) & p_{n1}(\Delta t) & \cdots & p_{nm}(\Delta t)
\end{bmatrix}.
\]
(7)

Where

\[
p_{00}(\Delta t) = \Delta F_{f(t+\Delta t)}^{(n)}} \Delta f_{f(t)}^{(n)}} C_X(u_1, u_2, \ldots, u_n)
\]

\[
A_{f(t)}^{(n)}} \Delta f_{f(t)}^{(n)}} C_X(u_1, u_2, \ldots, u_n)
\]

And \( F(t) = 1 - \exp\left(-\int_0^t \lambda(x)dx\right) \), Sign \( \Delta \) represents difference operation, \( \Delta f(x) = f(x_i) - f(x_j) \). Similarly,

\[
p_{0r}(\Delta t) = \begin{bmatrix}
p_{00}(\Delta t) & \cdots & p_{0r}(\Delta t) & \cdots & p_{0m}(\Delta t) \\
p_{10}(\Delta t) & \cdots & p_{1r}(\Delta t) & \cdots & p_{1m}(\Delta t) \\
\vdots & \ddots & \ddots & \ddots & \vdots \\
p_{n0}(\Delta t) & \cdots & p_{nr}(\Delta t) & \cdots & p_{nm}(\Delta t)
\end{bmatrix}
\]

\[
\Delta f_{f(t)}^{(n)}} \Delta f_{f(t)}^{(n)}} C_X(u_1, u_2, \ldots, u_n)
\]

\[
p_{10}(\Delta t) = P(Y_i \leq \Delta t) = G(\Delta t).
\]

\[
p_{20}(\Delta t) = P(Y_i + Y_j \leq \Delta t)
\]

\[
\sum_{i=1}^{n} \int_{0}^{\Delta t} \int_{0}^{\Delta t} \frac{\partial^2 C_f(v_1, v_2)}{\partial v_1 \partial v_2} \bigg|_{v_1 = G(y_i)} \prod_{j=1}^{\infty} \int_{0}^{\Delta t} g(y_j)dy_j dl_y
\]

\[
\prod_{i=1}^{n} \int_{0}^{\Delta t} g(y_i)dy_i \cdots dy_{n-k+1}.
\]

(9)

Here, \( G(y_i) = 1 - \exp\left(-\int_0^{y_i} \mu(x)dx\right) \), \( G(.) \) and \( g(.) \) are distribution function and density function of repair time \( Y_i \) for the failure component.

On the one hand, calculating (9) by numerical integral, another way is to decompose the \( r \)-integral into single integral on bi-Copulas by iteration and dimensionality reduction. The other element of \( \mathbf{P}(\Delta t) \) (the probability \( p_{ik}(\Delta t) \) of state \( h \) transit to state \( k \)) is

\[
p_{ik}(\Delta t) = P(Y_i + Y_k \leq \Delta t < Y_i + Y_{k-1}) = \int_{y_i+y_k+y_{k-1}}\int_{y_i+y_{k-1}} g_{h-k+1}(v_1, v_2, \ldots, v_{n-k+1}) \Delta f_{f(t)}^{(n)}} \Delta f_{f(t)}^{(n)}} C_X(u_1, u_2, \ldots, u_n)
\]

\[
\int_{y_i+y_k+y_{k-1}}\int_{y_i+y_{k-1}} g_{h-k+1}(v_1, v_2, \ldots, v_{n-k+1}) \Delta f_{f(t)}^{(n)}} \Delta f_{f(t)}^{(n)}} C_X(u_1, u_2, \ldots, u_n)
\]

\[
\prod_{i=1}^{n} \int_{0}^{\Delta t} g_i(y_i)dy_i \cdots dy_{h-k+1}.
\]

(10)

Note that when \( 1 \leq h < k \leq n \), \( p_{hk}(\Delta t) = 0 \).

Imitation on the above one step, calculate \( \mathbf{P}(\Delta t) \to Q \) with \( q_{ij} = \lim_{\Delta t \to 0} \frac{1 - p_{ij}(\Delta t)}{\Delta t} \) and

\[
q_{ij} = \lim_{\Delta t \to 0} \frac{p_{ij}(\Delta t)}{\Delta t}.
\]

One can solve the differential equations of system By means of Laplace transform under the Initial state vector \( \mathbf{p}(0) = [p_{00}(0), p_{10}(0), \ldots, p_{n0}(0)] \), and obtain the availability \( p_{00}(t) \) of system at \( t \) and the steady state availability \( p_{00}(\infty) \).

V. CALCULATION MODEL FOR TWO DIFFERENT UNITS PARALLEL REPAIRABLE SYSTEM

As the number of states increases rapidly, parallel repairable systems are complicated than in Series, but the \( n \) identical units and the composition of a mechanic of repairable systems, the analysis steps and methods ibid. not repeat them. A two different units and a parallel system is used to illustrate the it’s reliability analysis method, in which \( X_i \) and \( Y_i \) are not independent, and \( \lambda_i(t) \), \( \mu_i(t) \) need not to submit the exponential distributions.

Let \( \lambda_1(t), \lambda_2(t) \) be the hazard rates of two components respectively, \( \mu_1(t), \mu_2(t) \) are corresponding repair rates, the correlation structure between \( X_1 \) and \( X_2 \) is
Copula $C_{ij}(u_1,u_2) = C_{ij}(v_1,v_2)$ corresponds to $Y_1$ and $Y_2$. Assume the failure components immediately access to maintenance state, Obviously the system have five possible states.

The following is the state description, the state “0” represents the component 1 and 2 are all available, the component 2 is available, but component 1 is under maintenance; “3” represents component 1 is under maintenance and the 2 is waiting to maintenance; “4”represents component 2 is under maintenance and the 1 is waiting to maintenance; thus the system fault.

So one step state transition matrix $P(\Delta t)$ for this repairable systems in minimal time difference $t\rightarrow t+\Delta t$ was

$$P(\Delta t) = 
\begin{pmatrix}
    p_{00}(\Delta t) & \cdots & p_{04}(\Delta t) \\
    \vdots & \ddots & \vdots \\
    p_{40}(\Delta t) & \cdots & p_{44}(\Delta t)
\end{pmatrix}$$

$$p_{00}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{01}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{02}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{03}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{04}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{10}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{11}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{12}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{13}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{14}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{20}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{21}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{22}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{23}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{24}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{30}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{31}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{32}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{33}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{34}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{40}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{41}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{42}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

$$p_{43}(\Delta t) = \Delta^t_{f_1(t+\Delta t)}(u_1,u_2) C_X(u_1,u_2)$$

Here, $F, G$ similarly represent the distribution functions of $X_i$ and $Y_i$, respectively.

By solving the differential equations of this system

$$\frac{d}{dt}[p_0(t), p_1(t), \cdots, p_4(t)] = [p_0(t), p_1(t), \cdots, p_4(t)]Q$$

$$p(0) = [p_0(0), p_1(0), \cdots, p_4(0)]$$

(11)

One can calculate the availability of such parallel system at time $t$

$$A(t) = \sum_{i=0}^{2} p_i(t).$$

VI. CONCLUSION

Tried to consider two type of the positive correlation between the working lives lengths and the repair time lengths of all the components of the repairable systems, Copula theories are introduced to the reliability calculation methods for repairable systems, solved the problem that how to calculate the reliability involving failure correlation. And all the time variables in these models were extended to the general continuous distributions. But in that case, it is somewhat difficult to solve the differential equations of repairable systems, so how to search for more simple and efficient numerical solution, it is worth further study. In addition, failure data and statistical methods are required to complete the Copula model selection and the estimation to correlation degree parameter $\theta$.

REFERENCES


