THE GIBBS PHENOMENON BOUNDS IN WAVELET APPROXIMATIONS

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ABSTRACT

Typical Gibbs phenomenon manifests itself by appearance of overshoots and undershoots around the jump discontinuities. It is well known that the first maximum and minimum of approximation of the \( \text{sgn} \) function by the truncated Fourier integral is 1.17898 and 0.9028 respectively, which divided to the jump discontinuity correspond to 8.95% the Gibbs overshoot and to 4.86% undershoot. We proved that the Gibbs overshoot of any integral wavelet transform is less than 8.95%; if into integral wavelet transform is defined by tight wavelet, then the Gibbs overshoot does not exceed 7.07%. When the integral wavelet transform is defined by the Shannon wavelet, the Gibbs overshoot and undershoot equals to 7.07% and 1.73%. We have found compactly supported orthogonal wavelet having filter length 10, that defines dyadic wavelet expansion having bigger the Gibbs overshoot then the one of the Fourier integral.

1. INTRODUCTION

The well-known Gibbs phenomenon was first discovered by Henry Wilbraham [1] in 1848, and after half century the famous physicist Josiah W. Gibbs [2] explained its presence in the output of the supposedly extremely accurate harmonic analyzer of Albert A. Michelson and S.W. Stratton in 1898 [3]. In lossy image compression the Gibbs phenomenon is undesirable and it manifests itself as a ripple type distortions near sharp edges. As an advantage of the Gibbs phenomenon we can mention its application for image enhancement and edges (contours) detection. Roerdink and Zwaan [4] had used appearance of the overshoots and undershoots for enhancement of image of the heart cross section. In Magnetic Resonance Imaging this technique helped to sharpen edges defining the affected area from the surrounding healthy tissue of the heart [4]. Bauer [5] applied the Gibbs phenomenon for more precise determination of a shock (or shocks) in wave propagation.

In this paper we made an attempt take a look at the Gibbs phenomenon in wavelet expansions or approximations that are widely used in digital sound, image and video processing.

2. THE GIBBS PHENOMENON IN INTEGRAL FOURIER AND WAVELET APPROXIMATIONS

The integral Fourier transform

\[ x^\wedge(f) = \int_{-\infty}^{\infty} e^{-2\pi ft} x(t) dt, \]

and the inverse Fourier transform

\[ x(t) = \int_{-\infty}^{\infty} e^{2\pi ft} x^\wedge(f) df, \]

are well defined as square integrable functions for square integrable over \((-\infty, \infty)\) signals \(x(t)\) and \(x^\wedge(f)\). The \textit{signum} function \(\text{sgn}(t)\),

\[ \text{sgn}(t) = \begin{cases} 1, & t > 0, \\ -1, & t < 0, \end{cases} \]

which is a standard function in the Gibbs phenomenon analysis, has the Fourier transform

\[ x^\wedge(f) = \frac{1}{2\pi} \frac{1}{i\pi} \frac{1}{f}, \]

in the sense of distributions.

The truncated Fourier integral of \(\text{sgn}(t)\), with the integral in (2) being forced to have \textit{finite} symmetric limits of integration, is

\[ \text{sgn}_\lambda(t) = \frac{2}{\pi} \int_0^\lambda \frac{\sin 2\pi tf}{f} df = \frac{2}{\pi} \text{Si}(2\pi \lambda t), \]

where \text{Si} is the \textit{sine integral} function. The first maximum of \(\text{sgn}_\lambda(t)\) is at \(t_1 = \frac{1}{2\lambda}\) whose magnitude is

\[ \frac{2}{\pi} \text{Si}(\pi) \approx \frac{2}{\pi} \cdot \pi \cdot (1.17898) = 1.17898. \]
After normalization to the jump of sgn(t) at t = 0, we will have the overshoot of
\[ \frac{\text{sgn}_\lambda(1)}{2\lambda} - \frac{\text{sgn}(\frac{1}{2\lambda})}{2} \approx 0.0895. \]
So, the overshoot of the truncated Fourier representation of sgn function is close to 8.96%. The next extreme of sgn_\lambda(t) is a minimum at t = \frac{1}{\lambda} which gives the undershoot value (1 - 0.90282)/2 = 0.04859 or about 4.88%.

Relative to a square-integrable function \( \psi \), the integral wavelet transform for \( x \in L^2(-\infty, \infty) \) we will define as
\[
x^\psi(t, f) := \int_{-\infty}^{\infty} x(\tau) \sqrt{|f|} \psi^*_f(\tau) d\tau,
\]
where \( t, f \in (-\infty, \infty) \) and \( \psi^*_f(\tau) = \psi(f(\tau - t)) \). This definition a bit differs from the ordinary one given in [6], science we use time (t) and frequency (f) variables instead of shift (b) and scale (a) variables.

**Definition 1** Let \( \psi \in L^2(-\infty, \infty) \) and
\[
C_\psi := \int_{-\infty}^{\infty} \frac{\psi^*(f)^2 |f|}{|f|} df.
\]
It is called that the wavelet \( \psi \) satisfies the invertability condition, if
\[ 0 < C_\psi < \infty. \tag{8} \]
If the wavelet satisfies the invertability condition, then the signal \( x \) can be reconstructed from its integral wavelet transform \( x^\psi \) by the inversion formula [6]
\[
x(t) = C^{-1}_\psi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} x^\psi(t,f) \sqrt{|f|} \psi^*_f(t) d\tau df.
\]

We are interested in the wavelet representation of the signum function (3) which is non square-integrable over interval \((-\infty, \infty)\).

**Lemma 1** If \( x = \text{sgn}(t) \) and \( I_\psi(t) = \int_{-\infty}^{t} \psi(\tau) d\tau \),
then
\[ \text{sgn}_\psi(t, f) = -2 \text{sgn}(f) \frac{I_\psi(-ft)}{\sqrt{|f|}} \]  \tag{10}
and
\[ \text{sgn}(t) = C^{-1}_\psi \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \text{sgn}(t,f) \sqrt{|f|} \psi^*_f(t) d\tau df. \tag{11} \]

To compare the Gibbs phenomenons of integral Fourier and Wavelet approximations of sgn(t) function, we will consider the symmetric with respect to frequency \( f \) truncations of the integral wavelet representation of the signum function
\[ \text{sgn}^\psi_{\lambda}(t) = C^{-1}_\psi \int_{-\lambda}^{\lambda} \int_{-\infty}^{\infty} \text{sgn}(t,f) \sqrt{|f|} \psi^*_f(t) d\tau df. \tag{12} \]

Using (11) we have that \( \text{sgn}^\psi_{\lambda}(t) \) pointwise converges to \( \text{sgn}(t) \), as \( \lambda \to +\infty \). The appearance of the Gibbs phenomenon in integral wavelet approximation of the signum function depends on the wavelet \( \psi \). It is well known [7] that the Gibbs phenomenon does not appear when the integral wavelet transform is defined by the Haar wavelet. We have proved that the Gibbs overshoot in any integral wavelet approximation of sgn(t) is always smaller than the corresponding overshoot of the Fourier integral.

**Theorem 1** If the wavelet \( \psi \) satisfies the invertability condition
\[ 0 < \int_{-\infty}^{\infty} \frac{\psi^*(f)^2 |f|}{|f|} df < \infty, \]
then for any \( \lambda > 0 \)
\[ \sup_{t>0} \text{sgn}^\psi_{\lambda}(t) < \frac{2}{\pi} \text{Si}(\pi) \approx 1.17898. \tag{13} \]
Moreover,
\[ \sup_{\psi \in C_\psi \in \lambda > 0} \left( \sup_{t>0} \text{sgn}^\psi_{\lambda}(t) \right) = \frac{2}{\pi} \text{Si}(\pi). \]

Unfortunately paper size limitations do not allow us to present even sketch of proof of this and following statements.

In practical applications there arises often problem to recover any finite-energy signal \( x \) from its integral wavelet transform \( x^\psi \) using only the frequency samples \( f = \pm 2^j \), \( j \in \mathbb{Z} \). For this problem to have a solution, the basic wavelet function must satisfy more restrictive dyadic invertability condition [8].

**Definition 2** A function \( \psi \in L^2(-\infty, \infty) \) is called a dyadic wavelet if there exists two positive constants \( A \) and \( B \), with \( 0 < A < B < \infty \), such that
\[ A \leq \sum_{j=-\infty}^{\infty} |\psi^*(2^j f)|^2 \leq B, \hspace{1em} \text{a.e. on } (-\infty, \infty). \tag{14} \]
Relation (14) is called the dyadic invertability condition. If in (14) \( A = B \), i.e.
\[ \sum_{j=-\infty}^{\infty} |\psi^*(2^j f)|^2 = A \hspace{1em} \text{a.e.,} \tag{15} \]
then the wavelet \( \psi \) is called a tight dyadic wavelet.
The upper bound of the Gibbs phenomenon in tight dyadic wavelet expansions is a bit smaller than the one in Fourier integral.

**Theorem 2** Let $\psi$ is any tight dyadic wavelet. Then
\[
\sup_{t>0} \text{sgn}^\lambda_\psi(t) \leq \frac{2}{\pi} \int_{-1}^{0} \text{Si}(2\pi a_1 2^k) d\xi \approx 1.14138. \tag{16}
\]
Here $a_1 = 0.6956053$ is the first positive root of $\text{Si}(2\pi a) - \text{Si}(\pi a) = 0$.

If $\psi$ is the Shannon wavelet
\[
\psi(t) = 2 \frac{\sin(2\pi t)}{2\pi t} - \frac{\sin(\pi t)}{\pi t}, \tag{17}
\]
then
\[
\sup_{t>0} \text{sgn}^\lambda_\psi(t) = \frac{2}{\pi} \int_{-1}^{0} \text{Si}(2\pi a_1 2^k) d\xi \approx 1.14138. \tag{18}
\]
and
\[
\inf_{t>\frac{\pi a_1}{2\pi}} \text{sgn}^\lambda_\psi(t) = \frac{2}{\pi} \int_{0}^{1} \text{Si}(2\pi a_2 2^k) d\xi \approx 0.965457. \tag{19}
\]
Here $a_2 = 1.421418$ is the second positive root of $\text{Si}(2\pi a) - \text{Si}(\pi a) = 0$. The Gibbs overshoot and undershoot of the Shannon integral wavelet expansion, expressed in percents, are about $7.07\%$ and $1.73\%$ respectively.

3. **GIBBS BOUNDS OF DYADIC WAVELET EXPANSIONS**

Let's assume till the end of this paper that wavelets under consideration are real-valued. If $\psi$ is a tight dyadic wavelet with constant $A = 1$, then $\psi$ will be called normalized. Any normalized dyadic wavelet defines on $L_2(-\infty, \infty)$ the following expansion [8]:
\[
x(t) = \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} x^\lambda_j(r, 2^j) \sqrt{2^j} \psi_j(t) dr. \tag{20}
\]
A formal application of (20) for the signum function gives
\[
\text{sgn}(t) = -2 \sum_{j=-\infty}^{\infty} \int_{-\infty}^{\infty} \sqrt{2^j} \psi_j(-2^j t) \psi_j(t) dr = 2 \sum_{j=-\infty}^{\infty} I \psi \circ \psi(2^j t). \tag{21}
\]
To be (21) a valid representation of $\text{sgn}(t)$ we need in an additional assumption about $\psi$.

**Theorem 3** Let $\psi$ is a tight dyadic wavelet satisfying
\[
\int_{0}^{\infty} \frac{\left|\psi^\lambda(f)^2\right|}{f} \log f df < \infty. \tag{22}
\]
Then for any $t \in (-\infty, \infty)$
\[
\text{sgn}(t) = 2 \lim_{\mu_1, \mu_2 \to \infty} \sum_{j=-\infty}^{\mu_1-1} I \psi \circ \psi(2^j t). \tag{23}
\]
Using convergence conditions locality it is easy to understand that behavior around $t = 0$ of
\[
\text{sgn}_\mu^\lambda(2^j t) \tag{24}
\]
completely describes the Gibbs phenomenon of tight dyadic wavelet expansions. As in integral wavelet expansions, the Gibbs phenomenon does not appear in the Haar dyadic wavelet expansions. Let compare the Gibbs bounds of integral wavelet and dyadic wavelet expansions defined by the same tight wavelet.

**Theorem 4** If $\psi$ is normalized tight dyadic wavelet, then
\[
\lim_{\lambda \to -\infty} \frac{1}{2\ln 2} \sup_{t>0} \int_{-\lambda}^{\lambda} \int_{-\infty}^{\infty} \text{sgn}^\lambda_\psi(r, f) \sqrt{|f|} \psi_j^\lambda(t) dr df \leq 2 \sup_{t>0} \sum_{j<\mu} I \psi \circ \psi(2^j t), \tag{25}
\]
i.e. the Gibbs overshoot of integral wavelet expansion is always smaller than the one of dyadic wavelet expansion.

It is easy to verify that the bounds of the Gibbs phenomenon of the Shannon dyadic wavelet expansion of $\text{sgn}(t)$ coincide with those of the Fourier integral. Really, the representation of (24) in Fourier domain looks
\[
2 \sum_{j<\mu} I \psi \circ \psi(2^j t) = 2 \sum_{j<\mu} \int_{-\infty}^{\infty} \frac{|\psi^\lambda_\psi(2^j f)|^2}{2\pi f} e^{i\pi f t} df, \tag{26}
\]
and substitution of $\psi^\lambda$ into (26) gives
\[
\sum_{j<\mu} I \psi \circ \psi(2^j t) = \frac{1}{\pi} \sum_{j<\mu} \int_{2^{j-1}}^{2^j} \frac{\sin 2\pi f t}{f} df = \frac{1}{\pi} \text{Si}(\pi 2^\mu t). \tag{27}
\]
So, the partial sum (24) of the Shannon dyadic expansion of the signum function coincide with the one of the Fourier integral, which is taken at point $\lambda = 2^{\mu-1}$. Since the first maximum and minimum of (3) does not depend on $\lambda > 0$, the bounds of the Gibbs phenomenon

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of the Shannon dyadic expansion and the Fourier integral coincide.

The dyadic wavelet expansion gives a recipe of reconstruction of any finite-energy signal $x$ from its integral wavelet transform $x_0^\psi(t,f)$ using the frequency samples $f = \pm 2^j, j \in \mathbb{Z}$. The discrete wavelet expansion solves this problem using only values $x_0^\psi(2^{j-n}n, 2^j)$, $n, j \in \mathbb{Z}$.

Let suppose wavelet expansion are generated by Compactly Supported Tight Wavelet $\psi$ which is defined by two-scale relation with $2N$ filter length. Unfortunately only for $N = 1$ and in some exceptional cases for $N > 1$ an analytical form of compactly supported tight wavelets can be found. Therefore we used some numerical method for estimation of the Gibbs bounds of the wavelet expansions.

Let us give a formal definition of the global Gibbs overshoot and undershoot and their geometric mean.

**Definition 3** Let $\text{sgn}_\lambda(t)$ is a truncation of the $\text{sgn}(t)$ expansion, which pointwise converges to the signum function, as $\lambda \to \infty$. If, with any $\lambda$, $\text{sgn}_\lambda(t)$ is a continuous and odd function, then the quantities

$$G^+ := \lim_{\lambda \to \infty} \sup_{t > 0} (\text{sgn}_\lambda(t) - \text{sgn}(t))$$

and

$$G^- := \lim_{\lambda \to \infty} \inf_{t > 0} (\text{sgn}_\lambda(t) - \text{sgn}(t))$$

will be called the global Gibbs overshoot and the global Gibbs undershoot respectively. Here $t_1(\lambda)$ denotes the first positive root of the equation $\text{sgn}_\lambda(t) = 1$. The geometric mean value of the Gibbs overshoot and undershoot is defined as

$$G^\pm := \sqrt{G^+ G^-}.$$  

In percentage representation, $G^+, G^-, G^\pm$ are multiplied by 100 and divided by 2 (since the jump of $\text{sgn}(t)$ function at $t = 0$ equals 2).

The following table summarize numerically estimated extremal Gibbs bounds of integral and dyadic Wavelet Approximations (WA).

<table>
<thead>
<tr>
<th>Filter length $2N$</th>
<th>Integral WA</th>
<th>Dyadic WA</th>
</tr>
</thead>
<tbody>
<tr>
<td>4</td>
<td>3.35</td>
<td>3.44</td>
</tr>
<tr>
<td>6</td>
<td>5.12</td>
<td>6.60</td>
</tr>
<tr>
<td>8</td>
<td>5.79</td>
<td>8.46</td>
</tr>
<tr>
<td>10</td>
<td>6.24</td>
<td>9.08</td>
</tr>
<tr>
<td>12</td>
<td>6.46</td>
<td>9.79</td>
</tr>
</tbody>
</table>

4. CONCLUSIONS

It was shown that the supremum of the set of the Gibbs overshoots of integral wavelet expansions is equal to 8.95% of the size of the jump discontinuity. The one of integral wavelet expansions defined by tight wavelets equals 7.07%. If tight wavelet satisfy slightly stronger than invertibility condition, i.e.

$$\int_0^\infty \frac{|\psi^\lambda(f)|^2}{f} \ln |f| df < \infty,$$

then dyadic wavelet expansion of the signum function pointwise converges, and the Gibbs overshoot of integral wavelets expansion is always smaller than the one of dyadic wavelet expansion. There exists orthogonal wavelets, with filter length $2N = 10$, that define dyadic wavelet expansions with the Gibbs overshoot bigger than the one of the Fourier integral.

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5. REFERENCES


