Comparative study of the saturated sliding mode and anti-windup controllers

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Abstract—The saturation problem is one of the most common handicaps for applying linear control to real applications, especially the actuator saturation. This paper focuses on a comparative study between the classical anti-windup regulator and robust saturated sliding mode control. In the first step, we present a design methodology of SMC of a class of linear saturated systems. We introduce the structure of the saturation, then, we perform the design of the sliding surface as a problem of root clustering, which leads to the development of a smooth and non-linear control law that ensures to reach the sliding surface. The second step is devoted to present briefly the anti-windup controller technique. Finally, we use an example of a quarter of vehicle system to give simulation results.

Keywords: Variable Structure Control; Sliding Mode; Anti-windup; Satureted Systems; LMI.

I. INTRODUCTION

Most industrial processes operate in the areas characterized by many physical and technological constraints (saturation, limit switches...). The implementation of the control law designed without considering these limitations can have dire consequences for the system.

The problem of the control of saturated systems is a subject of great interest for applications. Used in early days, many rigorous design methods are available to provide guarantee properties on systems stability. Let us quote of these methods, the anti-windup design ([1], [2]...), and many other methods which introduce conditions on systems containing saturation functions ([1], [3], [4], [5], [6]...). In this paper, we are interested in the problem of sliding mode control of linear system with saturation on the entries. The sliding mode represents a very significant transitory mode for the Variable Structure Control (VSC) in terms of robustness. Early work in this area was mainly done by Soviet control scientists ([7], [8]...). In recent years, more and more research in this area has been done ([6], [9], [10]...).

This paper is organized as follows: in the beginning, we give a short introduction on the structure of the saturation constraint reported on the control vector and its implementation in the system. We will then present a design procedure of robust saturated sliding mode control. To validate the theoretical concepts of this work, we treat an application of a quarter of vehicle system where we will highlight a comparison between the sliding mode control and the anti-windup controller.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

Let us consider that the structure of the saturation constraint is described by figure 1:

![Structure of the saturation constraint](image-url)

\[ sat(u) = \begin{cases} 
U_{sat} & \text{if } u_i > U_{sat} \\
& \text{if } -U_{sat} < u_i < U_{sat} \\
-U_{sat} & \text{if } u_i < -U_{sat} 
\end{cases} \quad \forall i = 1, ..., m \]

(2)

we can write

\[ sat(u) = \beta u \]

(3)

the elements of \( \beta \) are expressed as follows

\[ \beta_i = \begin{cases} 
\frac{U_{sat}}{u_i} & \text{if } u_i > U_{sat} \\
1 & \text{if } -U_{sat} < u_i < U_{sat} \\
\frac{-U_{sat}}{u_i} & \text{if } u_i < -U_{sat} 
\end{cases} \quad \forall i = 1, ..., m \]

(4)

The saturated system can be written as:

\[ \dot{x}(t) = Ax(t) + B\beta u(t) \]

(5)

with \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( x \in \mathbb{R}^n \) and \( u \in \mathbb{R}^m \).

Assumption: The pair \((A, B)\) is controllable, \( B \) has full rank \( m \), and \( n > m \).
The sliding mode occurs when the state reaches and remains in the surface given by:

$$S = \bigcap_{j=1}^{m} S_j = \{ x \in \mathbb{R}^{n} | Cx = 0 \}$$  \hspace{1cm} (6)

The sliding mode occurs when the state reaches and remains in the surface intersection $S$ of the $m$ hyperplanes, geometrically the subspace $S$ is the null space of $C$.

Differentiating with respect the time,

$$\dot{x} = CAx + CB\beta u = 0$$  \hspace{1cm} (7)

if $(CB\beta)^{-1}$ exists, then

$$u_{eq} = -(CB\beta)^{-1}CAx = -Kx$$  \hspace{1cm} (8)

with $K = (CB\beta)^{-1}CA$

$$\dot{x} = (I_n - \beta B(CB\beta)^{-1}C)Ax = A_{eq}x$$  \hspace{1cm} (9)

the dynamics $\dot{x}$ (equation 9) describes the motion on the sliding surface and depends only on the choice of $C$.

III. DESIGN OF THE SLIDING SURFACE

The canonical form used in Reference [11] for VSC design to select the gain matrix $C$ that gives a good and stable motion during the sliding mode.

By assumption, the matrix $B$ has full rank $m$; as a result, there exists an $(n \times n)$ orthogonal transformation matrix $T$ such that: $TB = \begin{bmatrix} 0 \\ B_2 \end{bmatrix}$, where $B_2$ is $(m \times m)$ and nonsingular. Note that the choice of an orthogonal matrix $T$ avoids inverting $T$ when transforming back to the original system. As the transformed state variable vector is defined as:

$$y = Tx$$  \hspace{1cm} (10)

the state equation becomes

$$\dot{y}(t) = Tx(t) = TAx(t) + TB\beta u(t)$$  \hspace{1cm} (11)

If the transformed state is partitioned as $\dot{y} = \begin{bmatrix} y_1^T \\ y_2^T \end{bmatrix}$; $y_1 \in \mathbb{R}^{n-m}$, $y_2 \in \mathbb{R}^m$

then

$$\begin{cases} \dot{y}_1(t) = A_{11}y_1(t) + A_{12}y_2(t) \\ \dot{y}_2(t) = A_{21}y_1(t) + A_{22}y_2(t) + \beta B_2u(t) \end{cases}$$  \hspace{1cm} (12)

Since the sliding condition is

$$Cx = CT^Ty = 0$$  \hspace{1cm} (13)

with

$$TAT^T = \begin{bmatrix} A_{11} & A_{12} \\ A_{21} & A_{22} \end{bmatrix}, CT^T = \begin{bmatrix} C_1 \\ C_2 \end{bmatrix}$$  \hspace{1cm} (14)

We can the new defining sliding condition

$$C_1y_1 + C_2y_2 = 0$$  \hspace{1cm} (15)

Assumption: $CB$ is non-singular then $C_2$ must be non-singular.

The sliding mode condition becomes

$$y_2 = -C_2^{-1}C_1y_1 = -Fy_1$$  \hspace{1cm} (16)

with $F = C_2^{-1}C_1$ being an $(m \times (n-m))$ matrix. The sliding mode is then governed by the equations

$$\begin{cases} \dot{y}_1 = A_{11}y_1 + A_{12}y_2 \\ \dot{y}_2 = -Fy_1 \end{cases}$$  \hspace{1cm} (17)

representing an $(n-m)^{th}$ order system with $y_2$ playing the role of a state feedback control. The closed-loop system will then have the dynamics $\dot{y}_1 = (A_{11} - A_{12}F)y_1$. This indicates that the design of a stable sliding mode requires the selection of a matrix $F$ such that $\dot{y}_1 = (A_{11} - A_{12}F)y_1$ has $(n-m)$ left half-plane eigenvalues. Performances are taken into account via root clustering of the closed-loop dynamic matrix in a region of the complex plane. The area $\Omega(\alpha, -q, r, \theta)$ considered here is defined in Figure 2, which ensures a minimum decay rate $\alpha < 0$, a minimum damping ratio $\xi = \cos \theta$, and for relative stability and speed limitation can be made to place the eigenvalues in a circle in the left half complex plane.

![Fig. 2. LMI Region: intersection of three elementary regions.](image)

Where $F$ must be selected, that those $(n-m)$ eigenvalues of the system are in the region $\Omega(\alpha, -q, r, \theta)$, it can be determined by using root clustering with LMI concept. $C$ is given by them:

$$C = [F \quad I_{m}]T$$  \hspace{1cm} (18)

IV. SATURATED CONTROL LAW DESIGN

Once the existence problem has been solved that is the matrix $C$ has been determined, attention must be turned to solving the reachability problem. This involves the selection of a feedback control function $u(x)$ which ensures that trajectories are directed towards the switching surface from any point in the state space. The control strategy used here will be derived from that of Reference [12] which originated from the work of Gutman [13], and it consists of the sum of a linear
control law \( u^L \) and a nonlinear part \( u^N \). The general form is:

\[
    u(x) = u^L(x) + u^N(x) = Lx + \rho \frac{N_x}{\|Mx\| + \delta} 
\]

where \( L \) is an \((n-m)\) matrix, the null spaces of the matrices \( N; M; \) and \( C \) are coincident, and \( \delta \) is a small positive constant to replace the discontinuous component by a smooth nonlinear function, yielding chattering-free system response.

Starting from the transformed state \( y \), we form a second transformation \( T_2 : \mathbb{R}^n \rightarrow \mathbb{R}^n \) such that

\[
    Z = T_2y = T_2x; \quad Z^T = [z_1^T \ z_2^T] 
\]

And \( z_1 \in \mathbb{R}^{n-m}; z_2 \in \mathbb{R}^m \)

where

\[
    T_2 = \begin{bmatrix}
    I_{n-m} & 0 \\
    F & I_m 
    \end{bmatrix} 
\]

\( T_2 \) is non-singular as its inverse is given by

\[
    T_2^{-1} = \begin{bmatrix}
    I_{n-m} & 0 \\
    -F & I_m 
    \end{bmatrix} 
\]

The new state variables are then

\[
\begin{align*}
    z_1 &= y_1 \\
    z_2 &= Fy_1 + y_2 
\end{align*} 
\]

and the transformed system equation becomes

\[
\begin{align*}
    \dot{z}_1 &= \sum_1 z_1 + \sum_2 z_2 \\
    \dot{z}_2 &= \sum_3 z_1 + \sum_4 z_2 + \beta B_2 u 
\end{align*} 
\]

with

\[
\begin{align*}
    \sum_1 &= A_{11} - A_{12} F \\
    \sum_2 &= A_{12} \\
    \sum_3 &= F \sum_1 - A_{22} F + A_{21} \\
    \sum_4 &= A_{22} + A_{12} F 
\end{align*} 
\]

In order to attain the ideal sliding mode, it is necessary to force \( z_2 \) and \( \dot{z}_2 \) to become identically zero. To this end, the linear control law part \( u^L \) is formulated as

\[
    u^L(z) = - (\beta B_2)^{-1} [\sum_3 z_1 + (\Sigma_4 - \Sigma_4^*) z_2] 
\]

where \( \Sigma_4^* \in \mathbb{R}^{m \times m} \) is any design matrix with stable eigenvalues. In particular, we may set \( \Sigma_4^* = \text{diag}(\mu_i) \) such that Re \((\mu_i) < 0 \) for \( i = 1 \) to \( m \). Transforming back into the original x-space yields

\[
    u^L(x) = Lx = - (\beta B_2)^{-1} [\sum_3 (\Sigma_4 - \Sigma_4^*)] T_2 T x 
\]

\[
    L = - (\beta B_2)^{-1} [\sum_3 (\Sigma_4 - \Sigma_4^*)] T_2 T 
\]

Before presenting the nonlinear control law part \( u^N \) letting the matrix \( P_2 \) denote the positive definite unique solution of the Lyapunov equation

\[
    P_2 \Sigma_4^* + \Sigma_4^* P_2 + I_m = 0 
\]

then \( P_2 z_2 = 0 \) if and only if \( z_2 = 0 \), and we may take

\[
    u^N = - \rho (\beta B_2)^{-1} P_2 z_2 
\]

Transforming back into the original x-space, we obtain

\[
    u^N = - \rho (\beta B_2)^{-1} [0 \ P_2] TT_2 x 
\]

since the existence of the nonlinear component is checked we can deduce the matrices \( N \) and \( M \)

\[
    N = - (\beta B_2)^{-1} [0 \ P_2] TT_2 
\]

\[
    M = [0 \ P_2] TT_2 
\]

V. ANTIWINDUP CONTROL

The windup is a phenomenon in which a response becomes unstable or oscillated when the control input is saturated by a limiter in a control system which has integral terms. The main cause of this phenomenon is surplus integration of the integral terms in the controller when the input saturation occurs. As a result, errors appear in the state variable of the controller and it causes a windup phenomenon.

The integral term can be separated from PI/PID controller because these structures are comparatively easy. Therefore a common practice in the PI/PID controller is to stop the integration part while the input saturation is occurring.

Anti-windup is a traditional approach to dealing with actuator saturation. The idea is to augment the closed-loop system that was designed without taking actuator saturation into consideration so that the negative effect of actuator saturation is weakened. Earlier works on anti-windup design try to minimize the effect of saturation in a direct way by reducing the difference between the input and output of the actuators (see, for example, [14], [15]).

There have been many suggested ways of implementing Anti-windup control schemes, one of the best known is the so-called back information anti-windup scheme whose block diagram is depicted in Figure 3.

As displayed in Figure 3, the anti-windup control scheme introduces a new feedback signal \( "e_t" \) (saturation error) defined as:

\[
    e_t = u - v 
\]
where \( v \) is the magnitude of the control action requested by the control system, whereas \( u \) is the magnitude of the same control signal coming out from the saturation element.

In Figure 3, \( T_t \) stands for a saturation time constant whose aim is to speed up the incoming of the anti-windup scheme. As a rule of thumb, normally \( T_t < T_i \).

Under saturated conditions the integral part \( I \) of the control system will be given by:

\[
I = \frac{K_p}{T_i} \int_0^t e(t) \, dt + \frac{1}{T_t} \int_0^t e_i(t) \, dt \tag{35}
\]

and

\[
V(t) = \frac{K_p}{T_i} \int_0^t e(t) \, dt + \frac{1}{T_t} \int_0^t e_i(t) \, dt \tag{36}
\]

Therefore, from the above equation, we see that the aim of the anti-windup control scheme is to modify the value of the integral control action in a way that leads to a faster recovery from saturation conditions.

It is clear that when no saturation problems occur \( v = u \), therefore \( e_t = 0 \) and the action of the anti-windup control scheme will be canceled recovering the conventional feedback control structure.

VI. NUMERICAL APPLICATION

We consider a two degree of freedom vibrating system with one actuator describes by figure 4. The state equation of the system is given by,[16]:

\[
\dot{X}(t) = Ax(t) + Bu(t) \tag{37}
\]

with

\[
A = \begin{pmatrix}
0 & 0 & 1 & 0 \\
0 & 0 & 0 & 1 \\
-\frac{k_1+k_2}{m_1} & \frac{k_1}{m_1} & -\frac{C_1+C_2}{m_1} & \frac{C_2}{m_1} \\
-\frac{k_2}{m_2} & \frac{k_2}{m_2} & -\frac{C_1+C_2}{m_2} & \frac{C_2}{m_2}
\end{pmatrix}
\]

Simulation is achieved under the following condition:

\[
m_1 = m_2 = 1, k_1 = k_2 = 1, C_1 = C_2 = 0.01, \quad -1 \leq u(t) \leq 1
\]

The initial condition is given by \( x_0 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}^T \).

The figure (5) represents the poles of the reduced order system in an area defined by \( \Omega(\alpha, -q, r, \theta) = \Omega(-0.7, 0, 2, \pi/8) \) in the complex plan.

After 3 iterations, the algorithm gives the stabilizing gain \( F \) of the reduced system

\[
F = \begin{bmatrix} -3.0734 & -1.5077 & 2.7477 \end{bmatrix}
\]

According to (18)

\[
C = \begin{bmatrix} 3.0734 & -1.5077 & -1.0000 & 2.7477 \end{bmatrix}
\]

Figure (6) and figure (7) present, respectively, the evolution of the control input and state variables ( -: Anti-windup controller, -: sliding mode control).
Figure (6) presents the evolution of the saturated control input. It’s clear that the two controllers are saturated and always inferior to its maximal value in the two cases, but we can check that the Anti-widup controller have a more transient mode and the convergence is more slowly than that of the first control, what proves that the robust stability of the SMC is checked.

Figure (7) compares displacements of $x_1$, $x_2$, $x_3$ and $x_4$ with saturation. It shows a typical stable sliding mode convergence of the system using sliding mode control. As consequence, the state variables dynamics of the system with anti-windup controller have a more transient mode and the convergence is more slowly than that of the first control.

To analyze the robustness of the control techniques to parametric variations, we repeated the same simulation in the previous case but we have introduced variations of damping coefficients.

Figure (8) and figure (9) present, respectively, the evolution of the control input and state variables of the uncertain system (-: Anti-windup controller, –: sliding mode control).

Figures (8) and (9) show that the anti-windup control is sensitive to parametric variations. There is a significant performance degradation, especially the emergence of a strong oscillation reaction.

We found that the performance of the anti-windup control is limited only to systems without uncertainty. Indeed, the results obtained from parametric variations are completely unacceptable in an application context. Contrarily two figures (8) and (9) show that the behavior of the system in both cases (ideal system or uncertain system)
are almost similar. This confirms the robustness of the sliding mode control.

VII. CONCLUSION

In this paper, we proposed a comparative study of the sliding mode and anti-windup controllers for linear time invariant saturated systems. The structure of the saturation constraint is reported on the control input and being of constant limitations in amplitude. The design of the sliding surface is formulated as a pole assignment of a reduced system in an LMI region. The non-linear saturated control scheme is introduced, will be ensure the elimination of the undesirable chattering phenomenon and ensure a stable sliding mode motion. After that, we briefly had the principle and results of anti-windup controller and we will use the same structure of saturation presented. To verify the performance of the proposed SMC, we presented the simulation results for two controllers, applied to the "Two degrees of freedom vibrating system with one actuator". Indeed these simulation results show that the anti-windup controller is stable and acceptable, but the convergence is slowly and it’s sensitive to parametric variations. The SMC can remove the transient mode had been the main defect of the anti-windup control, and has the better performance. Consequently, we verified that the proposed SMC had the better robustness performance than the anti-windup control.

REFERENCES