FLATNESS-BASED TRAJECTORY GENERATION FOR INDUCTION MACHINE CONTROL

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ABSTRACT
This article deals with the problem of trajectory generation and tracking for nonlinear systems. The principal problem considered concerns the generation of desired trajectories for differentially flat systems. In this work, two methods of trajectory generation are applied in case of induction machine control to solve the problem of trajectory tracking with good efficiencies. The first one is based on the polynomial functions and the second rests on the B-splines functions. The combined methods of PI controllers and trajectory planning give an improvement results in terms of trajectory tracking in the case of field-oriented control strategy.

Index Terms— Trajectory tracking, Differentially flat systems, Induction machine, Polynomial functions, B-splines functions, Field-oriented control

I. INTRODUCTION
The induction machine is the machine of choice in many industrial applications due to its reliability, ruggedness, and relatively low cost. Nevertheless, the structure of induction machine which is very complex (nonlinear system, multi-variable and highly coupled) makes its control complicated and requires complex control algorithms. Thanks to advances in information technology, power electronics and automation, it is interesting to consider advanced structures of control based on control system methods, in order to adjust the speed such as flatness-based control [8], [1], [14] and vector control [2]. Flat systems, as introduced by Fliess et al. [8], are dynamic systems which are linearizable to controllable nonlinear system by means of endogenous feedback. It is shown in [12] and [6] that in the case of induction machine model, all state variables and control inputs can be expressed in terms of flat output and its higher order derivatives making easy the resolution of the trajectory planning problem. Indeed, one of the first steps of flatness control is to generate a desired trajectory for flat outputs to completely determine all variables of flat system. Several methods of trajectory generation are offered to us due to the differential properties of flat outputs. In [5], a polynomial interpolation can be used to define the trajectories of flat outputs. In [3], a significant number of theoretical results as the robust calculated algorithms for B-splines were provided.

In this paper, the trajectory tracking problem is treated by the combined methods of PI (Proportional Integral) controllers and the trajectory planning using flatness in the case of field control. The field-oriented control considers a decoupling between the torque and flux of induction machine. However, this technique is very sensitive to changes in machine parameters which can directly affect the performances obtained by conventional PI controllers. Then, the main objective of this control approach is to guarantee the desired performances in fact of robustness facing to the parameter changes and load torque variation.

This paper is organized as follows. Section 2 is devoted to the flatness of induction motor model by presenting the flat outputs which makes it possible to simplify the process open loop control. The resolution of the trajectory generation problem is treated in section 3 using two different methods. Section 4 discusses the application of field-oriented control to the induction machine by explaining the choice of mark running. Finally, the simulation results are given in the section 5 showing the effectiveness of the proposed approach.

II. FLATNESS OF INDUCTION MACHINE
Differential flatness was originally introduced by Fliess et al. [8] in a differentially algebraic context and later using Lie-Bäcklund transformation [9]. Mathematically, a nonlinear system:
\[
\dot{x} = f(x, u)
\]
is differentially flat if we can find flat outputs:
\[
z = h_f(x, u, \ldots, u^{(p)})
\]
such that:
\[
x = \varphi(z, \dot{z}, \ddot{z}, \ldots, z^{(r)})
\]
\[
u = \phi(z, \dot{z}, \ddot{z}, \ldots, z^{(r)})
\]
z is called the flat output which can be or not the system output. As represented above in equation (3), the states and the input variables of the system are expressed in terms of the flat outputs and their higher derivatives.

For the induction machine, three kind of models can be considered depending on the reference axis [4]:

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• if the reference frame is fixed with respect to the stator: an electric system is obtained where the statoric quantities are purely alternatives.

• if the reference frame is fixed with respect to the rotor: the electrical signals are quasi-continuous. This reference frame is privileged due to the quasi-continuous electrical quantities while accessing to the mechanical position.

• if the reference frame is linked to turning field: a purely continuous system is obtained which is ideally adapted to the regulation.

In this article, we will study, firstly, the trajectory generation based on the concept of differential flatness using the nonlinear model of induction motor with respect to a fixed stator reference frame. The control of induction motor will be then treated when the reference frame is linked to turning field, in order to have constant quantities where it is easier to make the regulation.

For this, consider the following nonlinear model of the induction motor with respect to a fixed stator reference frame, extensively developed in [10]:

\[ J \frac{d\Omega}{dt} = \frac{pM}{L_r} (\phi_{rd} i_{sq} - \phi_{rq} i_{sd}) - C_s \]

\[ L_r \frac{d\phi_{rd}}{dt} = -R_r \phi_{rd} - L_r p\Omega \phi_{rq} + R_r M i_{sd} \]

\[ L_r \frac{d\phi_{rq}}{dt} = -R_r \phi_{rq} + L_r p\Omega \phi_{rd} + R_r M i_{sq} \]

\[ \sigma L_s \frac{di_{sd}}{dt} = \frac{M R_r}{L_r^2} \phi_{rd} + \frac{p M}{L_r} \Omega \phi_{rq} - \left( \frac{M^2 R_r + L_r^2 R_s}{L_r^2} \right) i_{sd} + v_{sd} \]

\[ \sigma L_s \frac{di_{sq}}{dt} = \frac{M R_r}{L_r^2} \phi_{rq} - \frac{p M}{L_r} \Omega \phi_{rd} - \left( \frac{M^2 R_r + L_r^2 R_s}{L_r^2} \right) i_{sq} + v_{sq} \]

where \( \phi_{rd} \) and \( \phi_{rq} \) denote the components of the rotor flux vector with respect to a fixed stator reference frame. Similarly, \( i_{sd} \) and \( i_{sq} \) denote the stator currents with respect to the reference frame. The rotor speed is denoted by \( \Omega \). \( J \) denotes the moment of inertia of the rotor, \( p \) is the number of pole pairs, \( R_r \) and \( R_r \) denote, respectively, the stator and rotor electric resistances. \( L_s \) and \( L_r \) are the auto-inductances of the stator and rotor circuits. \( M \) is the mutual inductance and the parameter \( \sigma \) is given by \( \sigma = 1 - \left( \frac{M^2}{L_r L_s} \right) \). The control inputs to the stator circuit are denoted by \( v_{sd} \) and \( v_{sq} \). The load torque is denoted by \( C_s \). All constant parameters are assumed to be known.

The expression:

\[ C_e = \frac{p M}{L_r} (\phi_{rd} i_{sq} - \phi_{rq} i_{sd}) \]

represents the torque produced by the motor.

For a reason of simplicity, the complex representation is considered to define the variables of the system model. Set \( \rho = |\phi_e| \) and define \( \rho \) as the angle such that \( \phi_e = p e^{j \rho} \) (with \( j^2 = -1 \)), therefore \( \delta \) is the angle of the rotor flux with respect to a fixed frame defined in the form [6]:

\[ \delta = \alpha + \rho \theta \]

\[ z = (\theta, \alpha) \] is then the flat output of the induction motor. This output has a physical sense which can simplify the control design in the open-loop.

The mechanical equation of induction motor defined in (4) becomes:

\[ \frac{d\Omega}{dt} = \frac{p}{J R_r} \rho^2 \dot{\alpha} - \frac{C_s}{J} \]

It should be noted that \( \dot{\alpha} \) is the slip speed defined only in the operation modes at constant speed [6]. From (7), \( \rho \) is written as follows:

\[ \rho = \sqrt{\frac{R_r (J \dot{\theta} + C_s)}{p \dot{\alpha}}} \]

Consequently,

\[ \phi_r = \rho e^{j \alpha} = A (\dot{\theta}, \alpha, \dot{\alpha}, C_s) \]

The input equations of induction machine are then defined, a function of the flat output, as follows:

\[ i_r = \frac{1}{p R_r} \left( \phi_r - L_r i_s \right) e^{j \rho} \]

\[ i_s = \frac{1}{M} (\phi_r - L_r i_s) e^{j \rho} \]

\[ \phi_s = L_s i_s + M e^{j \rho} i_r = D (\theta, \alpha, \dot{\alpha}, C_s, C_s) \]

\[ v_s = R_s i_s + \frac{d}{dt} (\phi_s) = E (\theta, \alpha, \dot{\alpha}, \ddot{\alpha}, C_s, C_s) \]

III. TRAJECTORY GENERATION

The approach of trajectory generation based on the differential flatness property allows to solve the problem of trajectory planning by avoiding the integration of the differential equations of the dynamics system. In this section, we determine, first of all, the desired trajectories of flat outputs based on the polynomial functions. Then, a traditional and flexible methodology, rests on the B-splines functions [7], [11], is proposed in order to compare the performances in terms of desired trajectory tracking with polynomial functions.

III-A. Polynomial functions

The reference trajectories of induction motor variables are given according to the saturation levels of supply voltages, to the magnetic state and to the acceptable maximum currents by the machine. The desired trajectory \( \alpha^d \) with constant speed must sliding compared to the position of the rotor to allow the phenomenon of induction. In [5], the desired trajectory \( \alpha^d \) is determined at constant nominal speed in order to minimize the effective value of stator current \( i_s \). The value of stator current \( i_s \) is minimal when the constant value of slip speed \( \dot{\alpha}_{opt} = \frac{R_r}{L_r} \). Thus, the trajectory \( \alpha^d \) is given in such way to obtain a desired trajectory of speed \( \dot{\alpha}^d \) as the form presented in the figure 1 [5]. The initial and final conditions are the following ones:

\[ \alpha^d (0) = 0 \quad \dot{\alpha}^d (t_r) = \dot{\alpha}_{opt} \quad \ddot{\alpha}^d (t_r) = 0 \]
where \( t_r \) is the time of transition from an initial state to a final state.

**Figure 1. Desired trajectory of \( \dot{\alpha}^d(t) \).**

It’s necessary to choose a polynomial is chosen, at least, of order three to satisfy these conditions. There remains a degree of freedom for the trajectory design, which enables us to impose an additional condition \( \dot{\alpha}^d(0) = \dot{\alpha}_0 \). The trajectory of \( \dot{\alpha}^d(t) \) is then obtained in the following polynomial form:

\[
\dot{\alpha}^d(t) = \begin{cases} 
\dot{\alpha}_0 \left( \frac{t}{t_r} \right)^3 + (\dot{\alpha}_{opt} - \dot{\alpha}_0) \left( \frac{t}{t_r} \right)^2 - \frac{1}{3} (\dot{\alpha}_{opt} - \dot{\alpha}_0) & 0 \leq t \leq t_r \\
\dot{\alpha}_0 \left( \frac{t}{t_r} \right) - 3 \left( \frac{t}{t_r} \right)^2 \left( \frac{t}{t_r} \right) - \frac{1}{3} (\dot{\alpha}_{opt} - \dot{\alpha}_0) & t \geq t_r 
\end{cases}
\]

(12)

It is then necessary to solve the problem of singularity corresponding to the null sliding by explicit the desired trajectory speed \( \Omega^d \) according to the trajectory \( \dot{\alpha}^d(t) \) in place of time. The trajectory of desired speed \( \Omega^d \) allowing to join two constant values, must satisfy the constraints with continuity and derivability imposed on the flat output. The initial and final conditions to verify are:

\[
\Omega^d(0) = 0 \quad \Omega^d(t_r) = \Omega_{nom} \quad \dot{\Omega}^d(t_r) = 0
\]

(13)

where \( \Omega_{nom} \) is nominal speed to join in steady state. The speed trajectory, represented in the figure 2 is defined in the following polynomial form:

\[
\Omega^d(t) = \begin{cases} 
\text{Poly}(t) & 0 \leq t \leq t_r \\
\Omega_{nom} & t \geq t_r 
\end{cases}
\]

(14)

such as the polynomial \( \text{Poly}(t) \) is given by:

\[
\text{Poly}(t) = f(\alpha^d(t)) = 21 \Omega_{nom} \left( \frac{\alpha^d(t)}{\alpha^d(t_r)} \right)^5 - 35 \Omega_{nom} \left( \frac{\alpha^d(t)}{\alpha^d(t_r)} \right)^6 + 15 \Omega_{nom} \left( \frac{\alpha^d(t)}{\alpha^d(t_r)} \right)^7
\]

(15)

**Figure 2. Desired trajectory of \( \Omega^d(t) \).**

### III-B. B-splines functions

The B-splines functions are defined as a base of space representation of polynomials by pieces, with \( T = [t_0, ..., t_n] \) a vector of \( \lambda + 1 \) real components with \( t_i \leq t_{i+1} \) for \( i \in \{0, 1, ..., \lambda - 1\} \). The vector \( T \) is called a nodal sequence made up by the nodes \( t_i \). If a node is represented \( k \)-time in the nodal sequence, \( t_i = t_{i+1} = ... = t_{i+k-1} \), it is said that the node is of multiplicity \( k \).

Basic functions \( B_{i,m} \) are defined by recurrence according to the relation:

\[
B_{i,0}(t) = \begin{cases} 
1 & 0 \leq t \leq t_{i+1} \\
0 & \text{else} 
\end{cases}
\]

\[
B_{i,m}(t) = \frac{t - t_i}{t_{i+m} - t_{i+1}} B_{i,m-1}(t) + \frac{t_{i+m} - t}{t_{i+m} - t_{i+1}} B_{i+1,m-1}(t)
\]

(16)

where \( m - 1 \) is the degree and \( m \) is the order of the basic functions such as the order is the number of control points which affect the curve parameter.

Consider a B-splines curve made up of \( n+1 \) points \( C_j \) for \( j \in \{0, 1, ..., n\} \) and of order \( m \). In general, a B-splines curve \( X(t) \) is defined by the following relation:

\[
X(t) = \sum_{j=0}^{n} C_j B_{j,m}(t)
\]

(17)

In the case of a flat system, the parametrization of flat outputs \( z_i(t) \) by B-splines functions is defined in the form:

\[
z_i(t) = \sum_{j=0}^{n} C_{i,j} B_{j,m}(t)
\]

(18)

As the trajectories of flat outputs defined by B-splines are \( \beta \)-times continuously derivable on the open one \( [t_0, t_f] \), the following relation is then obtained:

\[
z_i^{(\beta)}(t) = \sum_{j=0}^{n} C_{i,j} B_{j,m}^{(\beta)}(t)
\]

(19)

In our case, a B-splines trajectory describing the profile of acceleration \( \ddot{\Omega}(t) = \frac{d\Omega}{dt} \) is first of all defined as the
form presented in the figure 3. The trajectory of speed \( \Omega^d (t) \) is obtained by integrating \( \dot{\Omega}^d (t) \) trajectory (figure 3).

![Figure 3. Desired trajectory profile of \( \Omega^d (t) \).](image)

In the next section, a classical control approach is presented to track the planned desired trajectories.

**IV. FIELD-ORIENTED CONTROL**

The vector model of induction machine is described by a set of electric, magnetic and mechanical equations coupled between them. To avoid the introduction on the electrical equation formulation, a fixed reference is chosen with respect to a turning field. In our case study, one supposed that:

\[
\phi_{rq} = 0, \quad \phi_r = \phi_{rd}
\]  

(20)

The equations of induction machine become:

\[
\frac{d}{dt} \phi_r = -\frac{1}{T_r} \phi_r + \frac{M}{T_r} i_{sd}
\]

\[
\sigma L_s \frac{d}{dt} i_{sq} = -R_s i_{sq} - \frac{M}{T_r} \frac{d}{dt} \phi_r + \sigma L_s \omega_s i_{sq} + v_{sd}
\]

\[
\sigma L_s \frac{d}{dt} i_{sq} = -R_s i_{sq} - \frac{M}{T_r} \omega_s \phi_r - \sigma L_s \omega_s i_{sd} + v_{sq}
\]

\[
\omega_r = \frac{L_r}{T_r} i_{sq}
\]

\[
C_e = \frac{P}{L_r} \phi_r i_{sq}
\]

(21)

where \( \sigma = 1 - \frac{M^2}{T_r L_s} \) denote dispersion coefficient and \( T_r = \frac{L_r}{R_r} \) the rotor time-constant. Setting \( i_\phi = \phi_r / M \) the magnetizing current, it results, from the knowledge of the rotor constant \( T_r \) and stator currents \( i_{sd} \) and \( i_{sq} \), the following estimate function:

\[
\ddot{i}_\phi = \frac{1}{(1 + T_r s)} i_{sd}
\]

(22)

\[
\ddot{w}_r = \frac{1}{T_r} \dot{i}_\phi
\]

(23)

where \( s \) is the Laplace operator. Introduce the expression (22) clarifies the magnetizing current \( i_\phi \) according to \( i_{sd} \).

one obtains:

\[
v_{sd} = R_s (1 + (T_r + T_p) s + \sigma T_s T_p s^2) \dot{i}_\phi - \sigma L_s \omega_s \dot{i}_{sq}
\]

\[
v_{sq} = R_s (1 + \sigma T_s s) i_{sq} + L_s \omega_s ((1 - \sigma) \dot{i}_\phi + \sigma i_{sd})
\]

\[
C_e = p L_s (1 - \sigma) i_\phi i_{sq}
\]

where \( T_p = \frac{L_p}{R_p} \) is the stator time-constant. The terms \( \sigma L_s \omega_s \dot{i}_{sq} \), \( \sigma L_s \omega_s \dot{i}_{sd} \) and \( L_s (1 - \sigma) \omega_s \dot{\phi} \) correspond to the coupling terms. It’s not possible then to control the two sizes \( i_\phi \) and \( i_{sq} \) independently. For that, the system is decoupled in such that the voltages \( v_{sd} \) and \( v_{sq} \) are reconstituted by the adjustment voltages \( u_{sd} \) and \( u_{sq} \) and the coupling terms.

For each current loop, a PI controller is designed. It contains a proportional action which is used to regulate the speed and an integral action which is used to eliminate the static error between the regulated output variable and the desired set-point.

**V. SIMULATION RESULTS**

For our simulations, an induction motor with two pairs of poles is considered, on balanced three-phase network of sinusoidal voltages with effective value \( V_s \) and fixed frequency \( f \), having the characteristics given by the table 1.

In the case of trajectory generation with polynomial functions, we fixed, on the one hand, the constant \( \dot{\omega_0} = 30 \text{ rad/s} \) to define the reference trajectories of flat outputs and, on the other hand, we accelerate the motor until its nominal speed \( \omega_{nom} = 150 \text{ rad/s} \).

In the case of trajectory generation with B-splines functions, a B-splines base of order \( m = 5 \) is chosen and defined on a nodal sequence \( T \) dividing the time interval into 5 segments. Then, at \( t = 13 \text{ s} \), a load couple \( C_s \) of approximately \( 6 \text{ Nm} \) corresponding to the value \( 2.5 \text{ A} \) of the current \( i_{sq} \) is applied to the induction machine.

The block diagram for the vector control with trajectory planning by flatness, is given in the figure 4.

![Figure 4. Field-oriented control with flatness-based trajectory planning.](image)

The magnitude of the speed is reduced by \(-6.53 \text{ rad/s} \) at \( t = 13 \text{ s} \) while the load torque \( C_s \) is applied to machine. The obtained results of the induction machine study show that the field-oriented control is a robust tool for control allowing to obtain a good performances in the
load mode (damping of the load couple variation).
In the tracking case, the polynomial functions and the B-spline functions are used for planning trajectories for flat outputs. Comparing the results presented in the figure 5 with those given in figure 6, an improvement in terms of reference trajectory tracking is involved by using polynomial functions. Indeed, the trajectory tracking error magnitude in the case of polynomial functions is more reduced than those in the case of B-spline functions.

VI. CONCLUSION

In this paper, two types of trajectory generation are studied for tracking in motion control algorithm. The field-oriented control with the flatness-based trajectory generation represents the main of this work. The performances obtained in reference trajectory tracking as well as the simplicity of the controller, justify the choice of flatness-based approach. In order to illustrate the effectiveness of differential flatness in the resolution of trajectory planning problem, the integration of flatness concept in the field-oriented control is considered.
The flatness concept requires the determination of flat outputs of process. Then, the definition of desired trajectories of these flat outputs taking into account the continuity and the derivability conditions deduced from the model.
The field-oriented control based on traditional controllers of PI type present a robust approach of control with respect to external disturbances.
The combined methods of PI controllers and flatness property in the field-oriented control give an improvement results in terms of trajectory tracking and robustness with respect to disturbances effects.

Table I. Parameters value [5].

<table>
<thead>
<tr>
<th>Parameter</th>
<th>Value</th>
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<tbody>
<tr>
<td>$V_n$</td>
<td>220 V</td>
</tr>
<tr>
<td>$f$</td>
<td>50 Hz</td>
</tr>
<tr>
<td>$R_c$</td>
<td>2.61 Ω</td>
</tr>
<tr>
<td>$R_s$</td>
<td>4.287 Ω</td>
</tr>
<tr>
<td>$L_r$</td>
<td>0.368 H</td>
</tr>
<tr>
<td>$L_s$</td>
<td>0.404 H</td>
</tr>
<tr>
<td>$M$</td>
<td>0.368 H</td>
</tr>
<tr>
<td>$J$</td>
<td>$25.6 \times 10^{-3}$ kg.m²</td>
</tr>
<tr>
<td>$\sigma$</td>
<td>0.089</td>
</tr>
</tbody>
</table>

VII. REFERENCES


Figure 5. Field-oriented control for $t_r = 5$ s - case of polynomial functions.
Figure 6. Field-oriented control for $t_r = 5$ s - case of B-splines functions.