Delay-dependent stability criteria for neutral systems with a varying delay

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Abstract— In this paper, we consider the problem of delay dependent stability of neutral system with time varying delays. By using an augmented Lyapunov functional, a new delay dependent sufficient stability criterion is obtained in terms of Linear Matrix Inequalities LMIs.

The proposed approach involves neither free weighting matrices, nor any model transformation, and it is shown that these criteria can provide less conservative results than some existing ones. Numerical examples are considered to show the efficiency of the proposed stability approach.

Keywords—Time varying delays; stability; Neutral systems; interval time varying delays; linear matrix inequalities.

I. INTRODUCTION

Time delay is often encountered in many dynamic systems such as chemical processes, neural networks and long transmission lines in pneumatic systems [6]. Since the existence of time-delays may be a source of oscillation, instability or and bad system performance, so considerable attention has been devoted to the stability of systems with time delays. In the literature, Lyapunov–Razumikhin functional and Lyapunov–Krasovskii functional are widely used approaches for analyses of time-delay systems to obtain a delay-independent or a delay-dependent stability condition [1-3].

As for delay-dependent stability, many methods have been taken for deriving stability criteria. The system transformation technique was employed in [7], and inequality methods were proposed for bounding cross terms in the derivative of the Lyapunov functional by [8]. The descriptor system approach incorporated with inequality methods is provided to be effective in [9]. Since system transformation and bounding techniques may give rise to conservatism. Recently, a free weighting matrix method is proposed in [10], where neither system transformation nor bounding technique was involved and free weighting matrices were introduced to estimate the derivative of the Lyapunov functional.

Over the past decades, the range of time-varying delay considered is from 0 to an upper bound. Which is not practical, the range of delay may vary in a range for which the lower bound is not restricted to be 0. For this reason, the stability of the systems with such interval time-varying delays has attracted considerable attention [11].

In this paper we propose improved delay dependent stability criteria for linear neutral system with varying delay. By constructing a Lyapunov-krasovskii’s function, its time derivative is estimated with less conservatism. Finally, two numerical examples are shown to support that our results are less conservative than those of the existing ones.

II. SYSTEM DESCRIPTION AND PRELIMINARIES

Considering the following neutral system with time-varying Delay:

\[
\begin{aligned}
\dot{x}(t) &= A x(t) + A_h x(t - \tau(t)) + C \dot{x}(t - \tau(t)) \\
\dot{x}(t) &= \phi(t), \quad \dot{x}(t) = \varphi(t) \quad \forall t \in [-\tau_2, 0]
\end{aligned}
\]

Where \(x(t) \in \mathbb{R}^n\) is the system state vector, \(\phi(.)\) and \(\varphi(.)\) are continuously differentiable vector valued initial function of \(t, \ t \in [-\tau_2, 0]\). \(A, A_h\) and \(C\) are known real constant matrices with appropriate dimensions.

It assumed that the time delay \(\tau(t)\) is a time varying continuous function that satisfies:

\[
\tau_1 \leq \tau(t) \leq \tau_2
\]

and

\[
\tau(t) \leq \mu \leq \mu_0
\]

Where \(0 \leq \tau_1 \leq \tau_2\) and \(\mu\) are constant. Note that \(\tau_1\) may not be equal to 0.

Lemma1[4]. For any scalar \(\tau(t) \geq 0\) and any constant matrix \(Q \in \mathbb{R}^{n \times n}\), \(Q = Q^T > 0\), the following inequality holds:

\[
- \int_{t-\tau(t)}^t \dot{x}^T(s) Q \dot{x}(s) ds \leq \tau(t) \xi^T(t) \tau(t) x(t) + 2 \xi^T(t) \tau(t) x(t) - x(t - \tau(t))]
\]

Where
\[ \xi(t) = [x^T(t) x^T(t - \tau(t))] x^T(t - \tau(t)) \]
and V is a free-weighting matrix with appropriate dimensions.

**Lemma 2 [5].** For any constant matrix \( w > 0 \), scalars \( a < b \) and vector function \( w(s) : [a, b] \to \mathbb{R}^n \) such that the following integrations are well defined, then

\[
\left( b - a \right) \int_a^b w^T(s) W w(s) ds \geq \int_a^b \left[ \int_a^t w^T(s) ds \right] W \left[ \int_a^t w(s) ds \right] \theta \geq \int_a^b \left[ \int_a^b w^T(s) ds \right] \theta W \int_a^b w(s) ds \theta;
\]

\[
\left( b - a \right)^2 \int_a^b \left[ \int_a^b w^T(s) ds \right] \theta W \left[ \int_a^b w(s) ds \right] \theta \geq \int_a^b \left[ \int_a^b w^T(s) ds \right] \theta W \left[ \int_a^b w(s) ds \right] \theta;
\]

\[
\left( b - a \right)^2 \int_a^b \left[ \int_a^b w^T(s) ds \right] \theta W \left[ \int_a^b w(s) ds \right] \theta \geq \int_a^b \left[ \int_a^b w^T(s) ds \right] \theta W \left[ \int_a^b w(s) ds \right] \theta;
\]

### III. Stability Analysis

This section performs stability analysis of neutral systems with time varying delay. The following theorem presents delay dependent stability criteria for system (1).

**A. Theorem 1.**

Given a scalars \( 0 \leq \tau_1 \leq \tau_2 \) and, the system (1) with time varying delay \( \tau(t) \) satisfying (2) and (3) is asymptotically stable if there exist real \( n \times n \) matrices

\[
P = P^T > 0, \quad i = 1, 2, 3, 4, 5 \quad Q_j = Q_j^T \geq 0,
\]

\[
R_j = R_j^T \geq 0, \quad j = 1, 2, 3, 4 \quad S_k = S_k^T \geq 0,
\]

\( k = 1, 2 \) such that the following LMI holds:

\[
\begin{bmatrix}
\phi & \tau \xi \\
* & -Q_4
\end{bmatrix} < 0 \quad (4)
\]

\[
\begin{bmatrix}
\phi & \tau_1 \xi' \\
* & -Q_2
\end{bmatrix} < 0 \quad (5)
\]

With

\[
\Phi_{11} = R_3 + \tau_1^2 Q_3 + \tau_2^2 A^T Q_4 A + A^T R_4 A
\]

\[-4 \tau_1^2 S_1 + \tau_2^2 A^T S_2 A + P_{14} + P_{24}^T + P_{14}^T
\]

\[+ A^T P_{11} + 2 \tau_1 U_1 + 2 \tau_2 U_1^T + R_1 + A^T R_6 A
\]

\[
\Phi_{12} = P_{11} A_h + 2 \tau_1^2 T R_4 A_h + 2 \tau_2^2 A^T Q_4 A_h
\]

\[+ 2 \tau_1^2 A^T S_3 A_h + 2 \tau_2^2 T R_6 A_h
\]

\[
\Phi_{13} = -P_{14} + P_{15} + A^T P_{24}^T + A^T P_{12}
\]

\[− U_1 + U_2^T
\]

\[
\Phi_{14} = −P_{15}
\]

\[
\Phi_{15} = P_{11} C + 2 \tau_2 A^T T C + 2 \tau_1^2 A^T Q_4 C
\]

\[+ 2 \tau_1^2 A^T S_3 C + 2 \tau_2^2 T R_6 C
\]

\[
\Phi_{16} = P_{12}; \quad \Phi_{17} = P_{13}; \quad \Phi_{18} = A^T P_{14}
\]

\[
\Phi_{19} = A^T P_{15}
\]

\[
\Phi_{22} = −P_{24} + P_{25} − P_{24}^T + P_{25}^T + A^T R_4 A_h
\]

\[+ \tau_1^2 A^T S_3 A_h + 2 \tau_2^4 A^T S_3 A_h − (1 − \mu) R_5 + A^T R_6 A_h
\]

\[
\Phi_{23} = A^T P_{12}; \quad \Phi_{24} = A^T P_{13}
\]

\[
\Phi_{25} = 2 A^T S_3 C + \tau_1^2 A^T S_3 C + 2 \tau_2^4 A^T S_3 C + 2 A^T S_3 C
\]

\[
\Phi_{28} = A^T P_{14}; \quad \Phi_{29} = A^T P_{15}
\]

\[
\Phi_{33} = −R_3 + R_1 − P_{24} + P_{25} − P_{24}^T + P_{25}^T
\]

\[+ \tau_1^2 Q_1 + 2 \tau_1 V_1 + 2 \tau_2 V_1^T − U_2 − V_2^T
\]

\[
\Phi_{34} = P_{35} − P_{34}^T − P_{25} − V_1 + V_2^T
\]

\[
\Phi_{35} = C^T P_{12}; \quad \Phi_{36} = P_{24}; \quad \Phi_{37} = P_{23}
\]

\[
\Phi_{38} = −P_{44} + P_{45}^T; \quad \Phi_{39} = −P_{45} + P_{55}
\]

\[
\Phi_{44} = −P_{35} − P_{35}^T − R_1 − V_2 − V_2^T
\]

\[
\Phi_{45} = C^T P_{13}; \quad \Phi_{46} = P_{23}; \quad \Phi_{47} = P_{33}
\]

\[
\Phi_{48} = −P_{45} + P_{49} = −P_{55}
\]

\[
\Phi_{55} = C^T R_4 C + \tau_1^2 C^T T Q_4 C
\]

\[+ \tau_1^2 C^T S_3 C − (1 − \mu) R_6 + C^T R_6 C
\]

\[
\Phi_{58} = C^T P_{14}; \quad \Phi_{59} = C^T P_{15}
\]

\[
\Phi_{66} = R_2 − R_2 + \tau_1^2 Q_2; \quad \Phi_{68} = P_{24}; \quad \Phi_{69} = P_{25}
\]

\[
\Phi_{77} = −R_2; \quad \Phi_{78} = P_{34}; \quad \Phi_{79} = P_{35}
\]

\[
\Phi_{88} = −[Q_3 + 4 S_1]; \quad \Phi_{99} = −Q_1
\]

**Proof.** Choose a Lyapunov functional candidate to be
\[ V(t) = \sum_{i=1}^{4} V_i(t) \]

With
\[ V_i(t) = \mu^T(t) P \mu(t) \]

\[ V_2(t) = \int_{t-\tau_1}^{t} x^T(s) R_1 x(s) ds + \int_{t-\tau_2}^{t} x^T(s) R_2 x(s) ds + \int_{t-\tau_1}^{t} x^T(s) R_3 x(s) ds + \int_{t-\tau_2}^{t} x^T(s) R_4 x(s) ds \]

Considering the above inequalities (3), one can obtain
\[ V_2(t) \leq x^T(t) (R_1 + R_3) x(t) + \dot{x}^T(t) (R_2 + R_4) \dot{x}(t) \]

The time derivative of \( V_3(t) \) is
\[ \dot{V}_3(t) = \tau_1 x^T(t) (t - \tau_1) Q_x x(t - \tau_1) + \tau_2 x^T(t - \tau_2) Q_x x(t - \tau_2) \]

Using the lemma 1 we obtain
\[ \dot{V}_3(t) \leq \tau_1 x^T(t) (t - \tau_1) Q_x x(t - \tau_1) + \tau_2 x^T(t - \tau_2) Q_x x(t - \tau_2) \]

And
\[ \mu^T(t) = \begin{bmatrix} x(t) \quad x(t - \tau_1) \quad x(t - \tau_2) \\ \int_{t-\tau_1}^{t} x(s) ds \quad \int_{t-\tau_2}^{t} x(s) ds \end{bmatrix} \]

Doing the time derivative of \( V_1(t) \) along the trajectory of system (1), we have
\[ \dot{V}_1(t) = 2 \mu^T(t) P \mu(t) \]

Taking the time derivative of \( V_2 \) yields
\[ \dot{V}_2(t) = x^T(t) (R_3 + R_2) x(t) + \dot{x}^T(t) (R_4 + R_6) \dot{x}(t) - x^T(t - \tau_2) R_5 x(t - \tau_2) - x^T(t - \tau_2) \dot{x}(t - \tau_2) + x^T(t - \tau_1) (R_1 - R_3) x(t - \tau_1) + x^T(t - \tau_1) (R_2 - R_4) \dot{x}(t - \tau_1) - (1 - \dot{\tau}(t)) x^T(t - \tau(t)) R_5 x(t - \tau(t)) - (1 - \dot{\tau}(t)) \dot{x}^T(t - \tau(t)) R_6 \dot{x}(t - \tau(t)) \]

With
\[ V = \begin{bmatrix} 0 & V_1^T & V_2^T \\ U_1^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ U_2^T & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} \]

And we use the Jensen’s inequality, we have
\[-\tau_1 \int_{t-\tau_1}^{t} x^T(s)Q_3x(s)ds \leq \left( \int_{t-\tau_1}^{t} x^T(s)ds \right)^T Q_3 \left( \int_{t-\tau_1}^{t} x^T(s)ds \right) \quad (9)\]

\[-\tau_{12} \int_{t-\tau_{12}}^{t} x^T(s)Q_4x(s)ds \leq \left( \int_{t-\tau_{12}}^{t} x^T(s)ds \right)^T Q_4 \left( \int_{t-\tau_{12}}^{t} x^T(s)ds \right) \quad (10)\]

Substituting (7) to (10) we get

\[\dot{V}_4(t) \leq \tau_{12}^2 x^T(t)Q_3x(t) + \tau_{12}^2 x^T(t)Q_4x(t) - \left( \int_{t-\tau_1}^{t} x^T(s)ds \right)^T Q_3 \left( \int_{t-\tau_1}^{t} x^T(s)ds \right) - \tau_{12}^2 \xi^T(t)Q_3^{-1} \xi(t) + 2\tau_{12} \xi^T(t)Q_3^{-1} \xi(t) + \tau_{12}^2 \xi^T(t)Q_4^{-1} \xi(t) + 2\tau_{12} \xi^T(t)Q_4^{-1} \xi(t) \]

Calculating \(\dot{V}_4(t)\), we have

\[\dot{V}_4(t) = \tau_1^4 x^T(t)S_1x(t) - 2\tau_1^2 \int_{t-\tau_1}^{t} x^T(s)S_1x(s)ds \theta \]

We use the lemma:

\[-2\int_{t-\tau_1}^{t} x^T(s)S_1x(s)ds \theta \leq -4(\int_{t-\tau_1}^{t} x(s)ds)^T S_1 \int_{t-\tau_1}^{t} x(s)ds \theta \]

\[\leq -4(\int_{t-\tau_1}^{t} x(s)ds)^T S_1(\tau_1^4 x(t) - \int_{t-\tau_1}^{t} x(s)ds) \]

We get

\[\dot{V}_4(t) \leq \tau_1^4 x^T(t)S_1x(t) - 4(\tau_1^4 x(t) - \int_{t-\tau_1}^{t} x(s)ds)^T S_1(\tau_1^4 x(t) - \int_{t-\tau_1}^{t} x(s)ds)\]

Finally, we have

\[V(t) = \xi^T(t) (\phi + \tau_{i2}^2 Q_3^{-1} \xi(t) + \tau_{i2}^2 Q_4^{-1} U^T )\xi(t)\]

Since \(\tau_{i2}^2 Q_3^{-1} \xi(t) + \tau_{i2}^2 Q_4^{-1} U^T\) is a convex combination of the matrices \(\tau_{i2}^2 Q_3^{-1} \xi(t)\) and \(\tau_{i2}^2 Q_4^{-1} U^T\) can be handled via two corresponding boundary LMIs

\[\phi + \tau_{i2}^2 Q_3^{-1} \xi(t) < 0 \quad (11)\]

\[\phi + \tau_{i2}^2 Q_4^{-1} U^T < 0 \quad (12)\]

By Schur Complement, the inequalities (11) and (12) are equivalent to the LMIs (4) and (5). Hence, system (1) is asymptotically stable. This ends the proof.

IV. NUMERICAL EXAMPLE

In order to show the effectiveness of the proposed methods, in this section, numerical examples are considered.

Example 1: Let consider the nominal neutral system (1) with the following parameters

\[A = \begin{pmatrix} -0.9 & 0 \\ 0.1 & -0.9 \end{pmatrix}, A_b = \begin{pmatrix} -1.1 & -0.2 \\ 0.1 & -0.9 \end{pmatrix}\]

\[C = \begin{pmatrix} -0.2 & 0 \\ 0.2 & -0.1 \end{pmatrix}\]

Table 1 lists the comparison of the upper bound on the allowable time delay that guarantee the asymptotic of system (1) by different methods. It can be seen that our stability condition is less conservative than those results discussed in [2, 12, 13].

<table>
<thead>
<tr>
<th>Methods</th>
<th>Maximum delay allowed</th>
</tr>
</thead>
<tbody>
<tr>
<td>[12]</td>
<td>1.3718</td>
</tr>
<tr>
<td>[13]</td>
<td>1.8266</td>
</tr>
<tr>
<td>[2]</td>
<td>2.0190</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>10^{11}</td>
</tr>
</tbody>
</table>

For \(\tau_1 = 0\)

Example 2: Consider the neutral system in the form of (1) with

\[A = \begin{pmatrix} -2 & 0 \\ 0 & -0.9 \end{pmatrix}, A_b = \begin{pmatrix} -1 & 0 \\ -1 & -1 \end{pmatrix}\]

\[C = \begin{pmatrix} c & 0 \\ 0 & c \end{pmatrix}\]

Table 2:

The Maximum delay allowed with different c

<table>
<thead>
<tr>
<th>c</th>
<th>0</th>
<th>0.1</th>
<th>0.5</th>
<th>0.9</th>
</tr>
</thead>
<tbody>
<tr>
<td>[13]</td>
<td>4.47</td>
<td>3.58</td>
<td>1.46</td>
<td>0.32</td>
</tr>
<tr>
<td>[14]</td>
<td>4.47</td>
<td>4.35</td>
<td>3.67</td>
<td>1.41</td>
</tr>
<tr>
<td>[15]</td>
<td>5.30</td>
<td>5.21</td>
<td>4.20</td>
<td>1.49</td>
</tr>
<tr>
<td>Theorem 1</td>
<td>10^{10}</td>
<td>3\times10^9</td>
<td>10^9</td>
<td>10^2</td>
</tr>
</tbody>
</table>

In table 2, the results of the Maximum delay allowed for guaranteeing stability are compared with the previous results. It can be shown that the proposed stability criterion for this system gives larger upper bound of time delay.
V. CONCLUSION
In this paper, delay-dependent stability criteria have been developed for systems with interval time-varying delay by applying an augmented functional Lyapunov–Krasovskii. To validate this fact, the proposed results is comparing with the results available in the existing literature, it is shown that the derived criteria are less conservative.

REFERENCES


