Optimal expansion of linear system using generalized orthogonal basis

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Abstract – This paper proposes a new method for determining the optimal pole of generalized orthonormal basis (GOB) which leads to an optimal system model developed on such basis. This optimal pole is obtained by solving a set of non linear equations. The proposed procedure is formulated for single-input/single-output (SISO) systems. The proposed algorithm is tested on simulations and good performances in term of approximation and calculus time are obtained.

Key words – Identification, General Orthonormal bases filters, SISO system, MISO system, poles, parameters.

I. INTRODUCTION

Recently a significant interest is giving in developing models for estimation of SISO (single-input/single-output) systems using the so-called orthonormal basis functions. This interest is motivated essentially by the drastic reduction of the model parameter number. Among these functions, we note Laguerre and Kautz basis functions that are limited to the favourite case when the basis functions have only respectively one real pole and a pair of complex conjugate poles. However, when the system class is other, the Generalized Orthonormal bases (GOB) filters can describe any unknown system with few parameters. Contrarily to Laguerre and Kautz functions, the GOB functions depend of several poles. Such problem of GOB poles optimisation has interested some authors, Den Brinker and al [4] developed a method to recursively determine GOB poles. Malti and al [2],[3] used the gradient algorithm and Gauss-Newton algorithm to minimize output quadratic error. Khouaja and al [5] used a method based on the estimation of the dominant mode associated with a residual signal obtained by iteratively filtering the process to be modelled. In this paper, we propose a new procedure of GOB based on the resolution of a set of non-linear equations. This new technique is developed for the SISO (Single-input/single-output) systems.

II. PROBLEM FORMULATION

Consider a observable stable SISO system. The output and input Z transform Y(z) and U(z) are related as :

\[ Y(z) = G(z)U(z) \]  \hspace{1cm} (1)

where G(z) is a stable transfer function describing the system dynamics. Ninness and Gustafsson [1] showed that the method of estimating the dynamics G(z) could be examined by using general orthonormal bases (GOB) given as follow:

\[ G(z) = \frac{Y(z)}{U(z)} = \sum_{n=0}^{N} g_n B_n(z) \]  \hspace{1cm} (2)

where \( g = \{ g_n \} \) are the parameters to be estimated, N+1 is the model order and \( \{ B_n(z) \} \) is a set of transfer function rational in the forward shift operator z given by:

\[ B_n(z) = \sqrt{1-\xi_n^2} \prod_{k=0}^{n-1} \frac{1-\bar{\xi}_k z}{z-\bar{\xi}_k} \]  \hspace{1cm} (3)

with \( \xi_n \) is the pole of the GOB function and \( \bar{\xi}_n \) its conjugate. Let’s define the pole vector \( \xi = [\xi_0 \cdots \xi_N]^T \). As the parameters \( g_n \) and the function \( \{ B_n \} \) depend on the pole vector \( \xi \), (2) can be rewritten as :

\[ G(z) = \sum_{n=0}^{N} g_n(\xi)B_n(z,\xi) \]  \hspace{1cm} (4)

From (1) the system output can be given by :

\[ Y(z) = \sum_{n=0}^{N} g_n(\xi)X_n(z) \]  \hspace{1cm} (5)

with \( X_n(z) = \sum_{n=0}^{N} B_n(z,\bar{\xi}_n) U(z) \) for \( n = 0,\ldots,N \)

(3) and (5) lead to the network of GOB filters given by Fig 1.

The number N that fixes the number of the model coefficients \( g_n \) depend on the choice of the poles \( \xi_n \). In other words the closer the pole vector to the optimal one is, the smaller the number N is. Therefore, we focus on these components of the vector \( \xi \). In the next section we provide a state space representation for any kind of the linear invariant-invariant (LTI) bounded-input/bounded output (BIBO) stable SISO system modelled in GOB bases.
III. STATE SPACE REPRESENTATION

With respect to relation (5) the output $y(k)$ and the input $u(k)$ can be connected via the network sketched in Fig.1. By considering $x(k)$ ($i = 0, ..., N$) as state variables, the network of Fig.1 leads to a state space representation of the system. From this network, we hold the equations (6).

\[
\begin{align*}
\begin{cases}
  x_0(k+1) &= \xi_0 x_0(k) + \sqrt{1-\xi_0^2} u(k) \\
  x_1(k+1) &= \xi_1 x_1(k) + T_1(1-\xi_0) x_0(k) - T_1 \xi_0 \sqrt{1-\xi_0^2} u(k) \\
  x_2(k+1) &= \xi_2 x_2(k) + T_2(1-\xi_1) x_1(k) - T_2 T_1 \xi_0 \xi_1 \sqrt{1-\xi_0^2} u(k) \\
  &\vdots \\
  x_N(k+1) &= \xi_N x_N(k) + (1-\xi_{N-1}) T_N x_{N-1}(k) + \sum_{j=0}^{N-2} (-1)^{N-j+1} (1-\xi_j) \prod_{i=j}^{N-1} T_i \xi_i \xi_{i+1} x_j(k) + \\
  &\quad + (-1)^N \sqrt{1-\xi_0^2} \prod_{i=j}^{N-1} T_i \xi_i \xi_{i+1} u(k)
\end{cases}
\end{align*}
\]

where for $i=1, ..., N$ we have $T_i = \frac{\sqrt{1-\xi_i^2}}{\sqrt{1-\xi_{i+1}^2}}$.

On the other hand, the output $y(k)$ can be written as

\[
y(k) = g_0 x_0(k) + g_1 x_1(k) + \cdots + g_N x_N(k) \quad (7)
\]

the state equation is given by:

\[
\begin{align*}
  \begin{cases}
    X(k+1) &= A X(k) + B u(k) \\
    y(k) &= C X(k)
  \end{cases}
\end{align*}
\]

with $X$ is the $(N+1)$-dimensional state vector, $A$ and $B$ are respectively the $(N+1)$ square matrix and the $(N+1)$ dimensional vector. The $(N+1)$-dimensional vector $C$ contains the coefficients $g_i$ ($i = 0, ..., N$) which correspond to the coefficients of the decomposition of the original transfer function on the GOB function.

These vectors and matrix are given by:
\[ X(k) = \begin{bmatrix} x_0(k) \\ x_1(k) \\ x_2(k) \\ \vdots \\ x_N(k) \end{bmatrix} \]

\[ B = \sqrt{1 - \xi_0^2} \begin{bmatrix} \xi_0 \\ T_z \xi_0 \xi_1 \\ \vdots \\ (-1)^{N+1} \prod_{i=0}^{N-1} T_z \xi_{i-1} \end{bmatrix} \]

\[ C = \begin{bmatrix} g_0 & g_1 & g_2 & \cdots & g_N \\ a_{1,1} & 0 & 0 & \cdots & 0 \\ a_{2,1} & a_{2,2} & 0 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N+1,1} & a_{N+1,2} & a_{N+1,3} & \cdots & a_{N+1,N+1} \end{bmatrix} \]

\[ A = \begin{bmatrix} a_{x,1} & a_{x,2} & a_{x,3} & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ a_{N+1,1} & a_{N+1,2} & a_{N+1,3} & \cdots & a_{N+1,N+1} \end{bmatrix} \]

And for \( \ell = 1, \ldots, N + 1 \) and \( q = 1, \ldots, N + 1 \) we have

\[
\begin{aligned}
    a_{\ell q} &= \xi_{\ell-1} & \text{for } \ell = q \\
    a_{\ell q} &= (1 - \xi_{\ell q})^2 \eta_q & \text{for } \ell = q + 1 \\
    a_{\ell q} &= (-1)^{q+1} (1 - \xi_{\ell q}) \prod_{i=q+1}^{\ell-1} T_z \xi_{i+1} & \text{for } \ell > q + 1
\end{aligned}
\]

As shown all the elements of the vector B and the matrix A depend only on the GOB poles \( \xi_i \). The truncating order N depends on the choice of \( \xi_i \) too. This order defines the dimension of the coefficient vector \( C \) to be estimated. In assuming that we have a record of \( H \) couples of inputs-desired outputs \( (d) \); the parameters vector \( C \) can be estimated by minimizing the least squares (LS) criterion:

\[
J(\xi) = \sum_{k=1}^{H} \left( y(k) - d(k) \right)^2 = (d - \Phi C)^T (d - \Phi C)
\]

Which leads to the well known LS solution:

\[
\hat{C} = (\Phi^T \Phi)^{-1} \Phi^T d
\]

Where \( \Phi \) is a regression matrix of dimension \((H,N+1)\)

The matrix \( \Phi \) depends on the choice of the GOB poles, the number of GOB filters \((N+1)\) and the horizon of observation \( H \).

**IV. OPTIMIZATION PROCEDURE**

For selecting the poles of a GOB, we assume that \( H \) couples of measurements \( \{u(k), y(k)\}_{k=0}^{H-1} \) and \( N+1 \) poles grouped in the set \( \xi = \{\xi_0, \ldots, \xi_N\} \) are available. Let us consider the real poles case. The poles are selected in the segment \([-1,1]\) to guarantee the model stability. From the network of figure 1, the output \( Z \)-transform \( Y(z) \) can be written as:

\[
Y(z) = g_0 \frac{1 - \xi_0^2}{z - \xi_0} U(z) + g_1 \frac{1 - \xi_1^2}{z - \xi_1} U(z) + \cdots + \frac{1 - \xi_N^2}{z - \xi_N} U(z)
\]

Thus

\[
\prod_{n=0}^{N} (z - \xi_n) Y(z) = g_0 \frac{1 - \xi_0^2}{z - \xi_0} \prod_{n=0}^{N} (z - \xi_n) U(z) + \cdots + \frac{1 - \xi_N^2}{z - \xi_N} \prod_{n=0}^{N} (z - \xi_n) U(z)
\]

That can be written as:

\[
F(z) Y(z) = \tilde{A}(z) U(z)
\]

where

\[
\tilde{A}(z) = \prod_{n=0}^{N} (z - \xi_n)
\]

and \( \tilde{H}(z) \) is the second member of (11). The polynomials \( F(z) \) and \( \tilde{H}(z) \) satisfies:

\[
F(z) = z^{N+1} - \sum_{i=1}^{N+1} f_i z^{N+1-i} \quad \text{and} \quad \tilde{H}(z) = \sum_{i=1}^{N} h_i z^i
\]

where \((f_i, h_i) \in \mathbb{R}^2\).
The problem of estimation of GOB poles can be resolved by determining the roots of the polynomial \( F(z) \). This need the knowledge of the parameters \( f_i, i = 1, \ldots, N + 1 \).

The \( Z^{-1} \) transform of (12) can be written as:

\[
y(k+N+1) - \sum_{i=1}^{N} (f_i y(k+N+1-i)) = \sum_{i=0}^{N} (h_i u(k+i))
\]

Our aim is to identify the coefficients \( f_i, i = 1, \ldots, N + 1 \) from \( H \) couples of measurements \( \{u(k), y(k)\}_{k=0}^{H-1} \). This identification allows us to determine the values of the poles.

Let's assume that \( H \) couples of measurements are available, for the variation of \( k \) from 0 to \( H-N-1 \) in (11), we obtain the following matrix form:

\[
Y = Q \theta
\]

Where

\[
Y = \begin{bmatrix} y(N+1) \\ \vdots \\ y(H) \end{bmatrix}, \quad \theta = \begin{bmatrix} \theta_1 \\ \theta_2 \end{bmatrix} \quad \text{with} \quad \theta_1 = \begin{bmatrix} p_0 \\ \vdots \\ p_N \end{bmatrix}, \quad \theta_2 = \begin{bmatrix} f_1 \\ \vdots \\ f_{N+1} \end{bmatrix}
\]

and

\[
Q = \begin{bmatrix} u(0) & \cdots & u(N) & y(N) & \cdots & y(0) \\ \vdots & \ddots & \vdots & \vdots & \ddots & \vdots \\ u(H-N-1) & \cdots & u(H-1) & y(H-1) & \cdots & y(H-N-1) \end{bmatrix}
\]

In this case, the most obvious and simplest scheme for estimating the parameter vector \( \theta \) is the ubiquitous least squares method. If the input is persistently exciting enough we can obtain the relation below:

\[
\theta_{opt} = (Q^T Q)^{-1} Q^T Y = \begin{bmatrix} \theta_{opt}^1 \\ \theta_{opt}^2 \end{bmatrix}
\]

so the knowledge of \( \theta_{opt}^2 \) allows to find the poles vector \( \zeta \) by using the routines of Matlab.

The optimization algorithm is summarized as follows:

**Algorithm 1**

1) Parameters
   - \( N+1 \) : truncation order.
   - \( H \) : couples of inputs-outputs

2) Computation
   a) Formation of the matrix \( Q \) according to (16)
   b) Estimate the parameters \( \theta_{opt}^2 \) according to (17)
   c) Evaluate the poles by using routines of Matlab

The identification performances are evaluated in terms of the normalized mean square error (NMSE):

\[
\text{NMSE}_{dB} = 10 \log_{10} \left( \frac{1}{H} \sum_{k=1}^{H} \frac{1}{2} \left( \frac{y(k) - d(k)}{y(k)} \right)^2 \right)
\]

### V. Numerical Simulation

- The first example of a SISO system is defined the following transfer function given by:

\[
G(z) = \frac{0.2 z^{-1} + 1.2 z^{-2} + 2.09 z^{-3}}{(1+0.2z^{-1})(1-0.8z^{-1})(1-0.5z^{-1})}
\]

The optimization procedure is based on input output observations, so by taking the response of the system to the random signal \( u(k) \) shown in Fig.2 we obtain the estimated GOB poles. When \( N=0 \) and \( N=1 \), the estimated poles gives a model degradation in terms of NMSE. The estimated poles are show in Fig.4 and Fig.5 and converges for \( N=0 \) and \( N=1 \) respectively to \( \zeta = [0.8784] \) and \( \zeta = [0.8095 \ 0.4071] \).

For \( N=2 \), our algorithm converges to \( \zeta = [0.4965 \ -0.2020 \ 0.8004] \) as shown in Fig.6.

![Fig.2. Input signal](image1.png)

![Fig.3. Output signal](image2.png)

![Fig.4. Estimated poles](image3.png)
The Fourier coefficients converge to $l_{10.2595, 2.3852, 2.2837}$ by applying the least squares method given by (9). With this truncation order $N=2$, the system and its resultant model response to $u(k)$ are shown in Fig.7.

The model gives a normalized mean squares error $\text{NMSE} = -34 \text{ dB}$. To check the performance of our method facing noisy signals, simulated white noise is added to the output of the system. When the signal to noise ratio (SNR) is 30 dB, the pole values converges to $\xi = [-0.211, 0.7987, 0.5106]$ and we obtain the model response shown in Fig.8.

To compare the proposed algorithm with the Gauss-Newton method that minimizes the Normalized Mean Square Error [2] for GOB poles estimation, we choose $N=2$. The GOB poles with the Gauss-Newton method converges to $\hat{\xi} = [0.915, 0.340, -0.843]$ with an initialization of this

consider actually a complex example given by its transfer function $F(z)$

$$F(z) = \frac{3.8z^3 - 3.71z^4 + 0.3995z^5 - 0.2439z^2 + 0.254z}{z^3 - 0.85z^4 - 0.555z^5 + 0.616z^3 - 0.0733z - 0.0612}$$  \hspace{1cm} (26)
poles equal to $[0.9, 0, -0.9]$. This method is very sensitive to the choice of the initial values of the GOB poles, and its convergence to the optimal values is not guaranteed. The convergence can lead to local minima. By applying our approach, the algorithm converges to the GOB poles values $\hat{\xi} = [0.9094, 0.4292, -0.8492]$, as shown in Fig.9. Our algorithm gave good performances in term of approximation and calculus times are obtained so it copes very well in case of adaptive identification scheme.

VI. CONCLUSION

This paper pointed out a new optimization approach for determining the optimum GOB poles. This approach is based on Newton’s iterative technique. The proposed algorithm can be used to design optimal GOB models for SISO systems that are stable linear and time invariant. This technique provides a better approximation than a higher order model with arbitrary chosen poles. Finally, this technique copes very well in case of adaptive identification scheme.

VII. REFERENCES


