Analysis and numerical solution of electromagnetic scattering from cavities

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Abstract — In this paper, we survey the mathematical study of the phenomenon of electromagnetic scattering from cavities. We consider time harmonic and transient fields, PEC and Impedance boundary conditions, fully embedded and over-filled cavities.

I. INTRODUCTION

The analysis of the electromagnetic scattering phenomenon induced by cavities embedded in an infinite ground plane is of high interest to the engineering community. Applications include the design of cavity-backed conformal antennas for civil and military use, the characterization of radar cross section (RCS) of vehicles with grooves, and the advancement of automatic target recognition. Due to its broad applications and challenge of solutions, the problem has been the focus of much mathematical research in recent years.

Here we provide a survey of mathematical research in this area. In addition we will describe the underlining mathematical formulation for this framework. Specifically, one seeks to determine the fields scattered by a cavity upon a given incident wave. The general way of approach involves decomposing the entire solution domain to two sub-domains via an artificial boundary enclosing the cavity: the infinite upper half plane over the infinite ground plane exterior to the boundary, and the cavity plus the interior region. The problem is solved exactly in the infinite sub-domain, while the other is solved numerically. The two regions are then coupled over the artificial boundary via the introduction of a boundary operator exploiting the field continuity over material interfaces.

II. PROBLEM SETTING

Fig. 1 depicts the geometry of an overfilled cavity. Let $\Omega \subset \mathbb{R}^2$ be the cross-section (cavity interior) of a $z$-invariant cavity in the infinite ground plane, and the infinite homogenous, isotropic region above the cavity as $\mathcal{U} = \mathbb{R}_+^2 \setminus \Omega$. Furthermore, let $B_R$ be a semicircle of radius $R$, centered at the origin and surrounded by free space, large enough to completely enclose the overfilled portion of the cavity. We denote the region bounded by $B_R$ and the cavity wall $S$ as $\Omega_R$, so that $\Omega_R$ consists of the cavity itself and the homogeneous part between $B_R$ and $\Gamma$. Let $\mathcal{U}_R$ be the homogeneous region outside of $\Omega_R$; that is, $\mathcal{U}_R = \{(r, \theta) : r > R, 0 < \theta < \pi\}$. Refer to

![Fig. 1. Problem Geometry – TM Polarization Depicted.](image)

The following is a formulation for a transient scattering problem in the 2D with impedance boundary condition. In this case, the magnetic
field $E$ is transverse to the $z$-axis so that $E$ and $H$ are of the form $E = (0,0,E_z)$ and $H = (H_x,H_y,0)$. In this case, the nonzero component of the total electric field $E_z$ satisfies the following boundary value problem:

$$-\Delta E_z + \varepsilon_r \frac{\partial^2 E_z}{\partial t^2} = 0 \quad \text{in } \Omega \cup \mathcal{U} \times (0,\infty),$$

$$\frac{\partial E_z}{\partial t} = -\frac{\eta}{\mu} \frac{\partial E_z}{\partial n} \quad \text{on } S \cup \Gamma_{\text{ext}} \times (0,\infty),$$

$$E_z |_{t=0} = E_0,$$

$$\frac{\partial E_z}{\partial t} |_{t=0} = E_{t,0} \quad \text{in } \Omega \cup \mathcal{U},$$

where $\varepsilon_r = \varepsilon / \varepsilon_0$ is the relative electric permittivity, $E_0$ and $E_{t,0}$ are the given initial conditions and $\eta = \sqrt{\mu / \varepsilon_r}$ is the normalized intrinsic impedance of the infinite ground plane. We are assuming that we have a non-dispersive material in the cavity, or that the permittivity is not a function of frequency, but could vary with respect to position. That is, we are assuming that the impedance is constant in the time domain.

We observe the scattered field $E_z^s$ solves:

$$-\Delta E_z^s + \frac{\partial^2 E_z^s}{\partial t^2} = 0 \quad \text{in } \mathcal{U} \times (0,\infty),$$

$$\frac{\partial E_z^s}{\partial t} + \frac{\eta}{\mu} \frac{\partial E_z^s}{\partial n} = -\left( \frac{\partial E_z^i}{\partial t} + \frac{\eta}{\mu} \frac{\partial E_z^i}{\partial n} \right),$$

on $\Gamma_{\text{ext}} \cup \Gamma \times (0,\infty),$

and also satisfies the appropriate radiation condition at infinity.

The homogeneous region $\mathcal{U}$ above the protruding cavity is assumed to be air and hence its permittivity is $\varepsilon_r = 1$. In $\mathcal{U}$, the total field can be decomposed as $E_z = E_z^i + E_z^s$ where $E_z^i$ is the incident field, and $E_z^s$ the scattered field.

We then decompose the entire solution domain to two sub-domains via an artificial semicircle, $\mathcal{B}_R$, which entirely encloses the overfilled cavity (refer to Fig. 2). These two sub-domains consist of the infinite upper half plane over the impedance plane exterior to the semicircle, denoted $\mathcal{U}_R$, and the cavity plus the interior region of the semicircle, denoted $\Omega_R$.

For this problem, as in [1], we will choose to use the Newmark scheme, an implicit time-stepping method that offers the advantage of stability. It is defined by the following: let $N$ be a positive integer, $T$ be the time interval, $\delta t = T / N$ be the temporal step size, and $t_{n+1} = (n+1)\delta t$ for $n = 0,1,2,...,N-1$. The following are approximations at $t = t_{n+1}$:

$$u_{n+1} \approx u, \quad \tilde{u}_{n+1} \approx \frac{\partial u}{\partial t}, \quad \tilde{u}_{n+1} \approx \frac{\partial^2 u}{\partial t^2}.$$

We further define $\gamma$ and $\beta$ as parameters to be determined to guarantee stability of the scheme, $\alpha^2 = \frac{1}{(\delta t)^2 \beta}$, and $\tilde{u}$ denotes predicted values.
Therefore, it can be shown that the scattered field \( u^{s,n+1} \) satisfies the following exterior problem:
\[
-\Delta u^{s,n+1} + \alpha^2 u^{s,n+1} = \alpha^2 \tilde{u}^{s,n+1} \quad \text{in } U_R,
\]
\[
u^{s,n+1}(R, \theta) = g(R, \theta) \quad \text{on } B_R,
\]
\[
\delta t \gamma \alpha^2 u^{s,n+1} + \eta \frac{\hat{u}^{s,n+1}}{\mu} = \delta t \gamma \alpha^2 \tilde{u}^{s,n+1} - \tilde{u}^{s,n+1} \quad \text{on } \Gamma_{\text{ext}},
\]
where \( g = u^{n+1} - u^{s,n+1} \) and the radiation condition is satisfied.

The corresponding variational formulation is to find \( u \in V \) such that:
\[
b_{TM}(u, v) = F(v) \quad \forall v \in V
\]
where
\[
b_{TM}(u, v) = \int_{\Omega_k} \nabla u \nabla v dxdy - \int_{\partial \Omega_k} T_R u v d\ell
+ \frac{\mu}{\eta} \Delta t \gamma \alpha^2 \int_S \tilde{u} v d\ell
+ \alpha^2 \int_{\Omega_k} \varepsilon v \tilde{u} dxdy,
\]
\[
F(v) = \int_{\Omega_k} J v d\ell - \int_{\partial \Omega_k} \Psi \tilde{u} v d\ell
+ \frac{\mu}{\eta} \Delta t \gamma \alpha^2 \int_S \tilde{u} v d\ell
- \frac{\mu}{\eta} \int_S \tilde{u} v d\ell + \alpha^2 \int_{\Omega_k} \varepsilon v \tilde{u} dxdy,
\]

**Theorem** The variational problem (6) is well-posed.

### III. CONCLUSION
We present a mathematical model for analyzing transient electromagnetic scattering induced by an overfilled cavity embedded in an impedance ground plane. We have established the well-posedness of the problem through a variational formulation, and this sets the foundation for numerical implementation through a hybrid finite element - boundary integral technique.

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**REFERENCES**