Globally attractive almost periodic solution of diffusion model with Beddington-Deangelis functional response

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Abstract—In this paper, a class of predator-prey model with Beddington-Deangelis functional response and diffusion is studied. By using Lyapunov function, the sufficient conditions are established for global attractivity of almost periodic solution of the system.

I. INTRODUCTION

Traditional Lotka-Volterra type predator-prey model with Holling type II functional response has received great attention from both theoretical and mathematical biologists, and has been well studied. But the response function in the model is prey-dependent only. So Arditi and Ginzburg[1] first proposed following ratio-dependent predator-prey model

\[
\begin{aligned}
x' &= x[a - bx] - \frac{cxy}{x + my} \\
y' &= -dy + \frac{fxy}{x + my} 
\end{aligned}
\tag{1.1}
\]

For the environmental parameters in ratio-dependent model (1.1) must be constants. Thus is rarely the case in real life. Meng Fan[2] incorporate the varying property of the model and studied the following system

\[
\begin{aligned}
x' &= x[a(t) - b(t)x] - \frac{c(t)xy}{x + m(t)y} \\
y' &= -d(t)y + \frac{f(t)xy}{x + m(t)y} 
\end{aligned}
\tag{1.2}
\]

However, ratio-dependent model has somewhat singular behavior at low densities which have been criticized and become the source of controversy. Therefore, many authors studied another functional response model, that is, Beddington-Deangelis functional response model[3,4,5], which has some of the same qualitative features as the ratio-dependent form but avoid some of the singular behaviors of the ratio-dependent at the low densities which have been the source of controversy. For example, literature [5] studied periodic and almost periodic solution of the system

\[
\begin{aligned}
x' &= x[a(t) - b(t)x] - \frac{c(t)xy}{\alpha(t) + \beta(t)x + \gamma(t)y} \\
y' &= -d(t)y + \frac{f(t)xy}{\alpha(t) + \beta(t)x + \gamma(t)y} 
\end{aligned}
\tag{1.3}
\]

. On the other hand, the predator-prey model with diffusion have been widely studied by many people either [6-12]. Literature [6] explore the conditions for the persistence and periodic solution of the model

\[
\begin{aligned}
x'_1 &= x_1[a_1(t) - b_1(t)x_1 - c_1(t)y_1] + e(t)(x_2 - x_1) \\
x'_2 &= x_2[a_2(t) - b_2(t)x_2 - c_2(t)y_2] + e(t)(x_1 - x_2) \\
y' &= y(-e(t) + d_1(t)x_1 + d_2(t)x_2 - f(t)y_2) 
\end{aligned}
\tag{1.4}
\]

Furthermore, almost periodic solutions of ecological model and its global attraction also attracted many author [13-22]. But to the best of our knowledge, few authors studied the predator-prey system with functional response and diffusion. So in this paper, we consider the following model with functional response and diffusion,

\[
\begin{aligned}
x'_1 &= x_1[a_1(t) - b_1(t)x_1 - \frac{c_1(t)y_1}{\alpha_1(t) + \beta_1(t)x_1 + \gamma_1(t)y_1}] + D_1(t)(x_2 - x_1) \\
x'_2 &= x_2[a_2(t) - b_2(t)x_2 - \frac{c_2(t)y_2}{\alpha_2(t) + \beta_2(t)x_2 + \gamma_2(t)y_2}] + D_2(t)(x_1 - x_2) \\
y' &= y(-d(t) + \frac{f_1(t)x_1}{\alpha_1(t) + \beta_1(t)x_1 + \gamma_1(t)y_1} + \frac{f_2(t)x_2}{\alpha_2(t) + \beta_2(t)x_2 + \gamma_2(t)y_2} - q(t)) 
\end{aligned}
\tag{1.5}
\]

where \(a_i(t), b_i(t), \alpha_i(t), \beta_i(t), \gamma_i(t), f_i(t), c_i(t), d_i(t), q(t), D(t)\) are positive continuous functions, \(y\) is predator and \(x\) is prey. \(x_1, x_2\) stand for the density of the population \(x\) in the two place respectively, \(D_i\) is the diffuse coefficient of the population \(x\). By using Lyapunov function, the sufficient conditions are established for global attraction of almost periodic solution of the system.

II. GLOBAL ATTRACTIVE OF ALMOST PERIODIC SOLUTION

Let \(R^+_3 = \{ (x_1, x_2, y) \in R^3 \mid x_1 \geq 0, x_2 \geq 0, y \geq 0 \}\).

Assume \(g(t)\) is a bounded continuous real function on \(R\), set \(g^m = \sup_{t \in R} g(t)\), \(g^l = \sup_{t \in R} g(t)\).

Lemma 2.1[6] \(R^+_3\) is a position invariant set with respect to system(1.5).
for convenience, we introduce the notations:
\[ M_1 = \max \{ \frac{a_i^m}{b_i}, i = 1, 2 \}, \alpha_0^m = \max \{ \alpha_i^m, i = 1, 2 \}, \]
\[ \alpha_i^t = \min \{ \alpha_i^t, i = 1, 2 \}, \beta_0^m = \max \{ \beta_i^m, i = 1, 2 \}, \]
\[ \beta_i^t = \min \{ \beta_i^t, i = 1, 2 \}, \gamma_0^m = \max \{ \gamma_i^m, i = 1, 2 \}, \]
\[ \gamma_i^t = \min \{ \gamma_i^t, i = 1, 2 \}, M_2 = \frac{f_{i^2}^{m} - d_{i}^m \beta_i^0}{d_{i}^m \gamma_i^0} M_1, \]
\[ m_1 = \min \{ \frac{a_i^m - c_i^m}{a_i^t (b_i^m + D_i^m)}, i = 1, 2 \}, \]
\[ m_2 = \frac{(f_{i^2}^{m} - d_{i}^m m_1 - \beta_i^m m_1 + \gamma_i^m M_2)q_i}{\alpha_0^m \beta_i^t + \gamma_i^m M_2}, \]
\[ \Delta (t, x, y) = \alpha(t) + \beta(t)x + \gamma(t)y, \quad \triangle_i (t, x, y) = \alpha_i(t) + \beta_i(t)x + \gamma_i(t)y, \quad (i = 0, 1, 2). \]

**Theorem 2.1** Assume \( m_1 > 0, m_2 > 0 \), then system (1.5) is persistence, that is to say, for each solution of system (1.5) with positive initial value \( \{ x_1(t), x_2(t), y(t) \} \), there exists \( T > 0 \), such that \[ m_1 \leq x_1(t) \leq M_1, m_1 \leq x_2(t) \leq M_1, m_2 \leq y(t) \leq M_2, \]
for \( t > T \).

**proof** Let \( V(t) = \max \{ x_1(t), x_2(t) \} \). Calculating the upper right Dini derivatives of \( V(t) \) along the solution of (1.5), \( i = 1, 2 \) we have
\[ D^+ V(t) \leq V(t) [a_i^1 - b_i^1 V(t)] \leq V(t) b_i^1 [a_i^m - b_i^1 - V(t)] \]
\[ \leq V(t) b_i^1 [M_1 - V(t)] \]

It follows that
(I) If \( V(0) \leq M_1 \), then \( V(t) \leq M_1, t > 0 \).
(II) If \( V(0) \geq M_1 \), let \( \alpha = \max \{ \alpha_i^m - b_i^1 (M_1) \} \), then there exists \( \epsilon > 0 \), such that \( V(t) > M_1 \) and \( D^+ V(t) < \alpha, t \in [0, \epsilon) \). Thus, there exists \( T_1 > 0 \), such that \( V(t) \leq M_1, t > T_1 \).

On the other hand, when \( t > T_1 \), we have
\[ y' \leq \left[ \frac{M_1}{\alpha^2 + \beta_2^t M_1 + \gamma_2^m y} - d_{i}^1 \right] \frac{f_{i^2}^m M_1}{Q_x^1} \]
\[ \leq \left[ \frac{f_{i^2}^m M_1}{Q_x^1} - d_{i}^1 \right] \frac{Q_x^1}{y} \]
\[ \leq \left[ \frac{f_{i^2}^m M_1}{Q_x^1} - d_{i}^1 \right] \frac{Q_x^1}{y} \]

We can choose an appropriate \( M_2 > M_2^* \), such that \( y \leq M_2^* < 0 \). Now by using the similar method (I), (II) used above, it is easy to show that there exists \( T_2 > T_1 \) such that \( y \leq M_2^* \) for \( t > T_2 \).

Furthermore, when \( t > T_2 \), we have
\[ x_i' \geq x_i [a_i^t - d_i^m b_i^m x_i - c_i^m M_2 - \frac{x_i b_i^m}{\alpha_i^t}] \]
\[ x_i [a_i^t - d_i^m b_i^m x_i - \frac{c_i^m M_2}{\alpha_i^t}] \]
\[ = x_i \left( a_i^t - d_i^m \right) \frac{\alpha_i^m}{\alpha_i^t} - c_i^m M_2 - b_i^m x_i \]

\[ = x_i b_i^m \left( \frac{a_i^t - d_i^m b_i^m}{\alpha_i^t} - c_i^m M_2 - x_i \right) \quad (i = 1, 2) \]

similarly, choose an appropriate \( M_1 \), such that \( 0 < m_1 < (a_i^t - d_i^m) \frac{a_i^m}{\alpha_i^t} - c_i^m M_2 - x_i \leq m_1, (i = 1, 2) \), which follows that \( x_i |_{x_i = m_1} > 0 \). Hence we obtain
(i) If \( x_i(0) \geq m_1 \), then \( x_i(t) \geq m_1, t \geq 0 \);
(ii) If \( x_i(0) < m_1 \) for \( x_i |_{x_i = m_1} > 0 \), then there exists \( T_i' \), such that \( x_i(t) \geq m_1, t > T_i' \). Take \( T = \max \{ T_i', T_2 \} \), we get that \( x_i(t) \geq m_1 \) when \( t > T' \).

On the other hand, for all \( t > T \), we have
\[ y' \geq \left[ d_i^m + \frac{f_{i^2}^m M_1}{Q_x^1} - \frac{q_i}{q_i} \right] \]
\[ = \left[ \frac{f_{i^2}^m M_1}{Q_x^1} - \frac{q_i}{q_i} \right] \]

similarly, take an appropriate \( m_2, 0 < m_2 < m_2^* \), such that \( y \leq m_2^* < 0 \). As a result, there exists \( T_3 > T \), such that \( y \geq m_2 \), when \( t > T_3 \). Set \( T = T_3 \), then \( m_1 \leq x_1(t) \leq M_1, m_1 \leq x_2(t) \leq M_1 \) and \( m_2 \leq y(t) \leq M_2, \) for all \( t > T \). That is, system (1.5) is persistence. This complete the proof.

In the following, we consider the existence and the global attractive of the almost periodic solution of system (1.5) with the positive almost periodic coefficient. First we introduce
\[ x' = f(t, x) \]
\[ x' = f(t, x), \quad y' = f(t, y) \]

**Lemma 2.2** Suppose that \( V(t, x, y) \) is defined on \( R^+ \times D \times D \) (or \( R^+ \times R^3 \times R^3 \)) satisfies
(1) \( a(||x - y||) \leq V(t, x, y) \leq b(||x - y||) \),
(2) \( V(t_1, x_1, y_1) - V(t_2, x_2, y_2) \leq L(||x_1 - x_2|| + ||y_1 - y_2||) \),
(3) \( V(t, x, y) \leq -CV(t, x, y) \),
where \( a(\gamma), b(\gamma) \) are increasingly positive continuous function; \( L, C \) are positive constant. Further assume that the solutions \( x(t) \) of (*) satisfies \( x(t) \in S \) for \( t \geq 0 \), \( S \) is a compact set, \( D \) is compact set and \( D \) union is a compact set and \( D \) union is a compact set. Then system (*) has a unique almost periodic solution \( p(t) \in D \), which is global attractive and \( p(t) \in S \), \( mod(p) \subset mod(f) \), especially, \( p(t) \in T \)-periodic solution of (*) when \( f(t, x) \) is \( T \)-periodic solution.

**Theorem 2.2** Assume that almost periodic system (1.5) satisfied the condition in Theorem 2.1 and,
\[ b_1(t) > \frac{c_1(t) \beta_1(t) \gamma_1(t) M_1 + f_1(t) \gamma_1(t) M_2 + f_1(t) \alpha_1(t)}{(\Delta_1(t, m_1, m_2)^2 + D_1(t) M_1 + D_2(t) \frac{1}{m_1})^2}
\]
\[ b_2(t) > \frac{c_2(t) \beta_2(t) \gamma_2(t) M_1 + f_2(t) \gamma_2(t) M_2 + f_2(t) \alpha_2(t)}{(\Delta_2(t, m_1, m_2)^2 + D_2(t) M_1 + D_1(t) \frac{1}{m_1})^2}
\]
\[ q(t) > \frac{c_1(t) \beta_1(t) \gamma_1(t) M_1 + c_1(t) \alpha_1(t)}{(\Delta_1(t, m_1, m_2)^2 + D_1(t) \gamma_1(t) M_1)}
\]

[4291]
\[
- \frac{f_2(t) \gamma_2(t) m_1}{\Delta_2(t, M_1, M_2)} - \frac{f_2(t) \gamma_2(t) m_1}{\Delta_2(t, M_1, M_2)} \quad \text{where } m_1, M_1, i = 1, 2 \text{ are defined in Theorem 2.1. Then there exists an unique almost periodic solution for almost periodic system (1.5).}
\]

**Proof** For \((x, y, z) \in \mathbb{R}^3\), we set \(\| (x, y, z) \| = |x| + |y| + |z|\). Let \(S = \{(x_1, x_2, y) | m_1 \leq x_1 \leq M_1, m_1 \leq x_2 \leq M_1, m_1 \leq y \leq M_2\}\), then \(S\) is compact set. According to Lemma 2.1, the solution of (1.5) with positive initial value is positive solution. Take \(X_i(t) = \ln x_i(t), Y(t) = \ln y(t), X_i^*(t) = \ln x_i^*(t), Y^*(t) = \ln y^*(t), i = 1, 2\).

Then system (1.5) can be written as
\[
\begin{align*}
\dot{X}_1 &= a_1(t) - b_1(t) e^{X_1} + \frac{c_1(t) e^{Y}}{\Delta_1(t, e^{X_1}, e^{Y})} + D_1(t) (e^{(X_2 - X_1)} - 1)
\dot{X}_2 &= a_2(t) - b_2(t) e^{X_2} - \frac{c_2(t) e^{Y}}{\Delta_2(t, e^{X_2}, e^{Y})} + D_2(t) (e^{(X_1 - X_2)} - 1)
\dot{Y} &= -d(t) + \frac{f_1(t) e^{X_1}}{\Delta_1(t, e^{X_1}, e^{Y})} + \frac{f_2(t) e^{X_2}}{\Delta_2(t, e^{X_2}, e^{Y})} - q(t) e^{Y} 
\end{align*}
\]

(2.1)

It is obviously that for each positive solution \(z(t) = (x_1(t), x_2(t), y(t))\) of system (3.1), \(Z(t) = (X_1(t), X_2(t), Y(t)) = (\ln x_1(t), \ln x_2(t), \ln y(t))\) is a solution of system (2.1); in verse, for every solution \(Z(t) = (X_1(t), X_2(t), Y(t))\) of system (2.1), \(x(t) = (\exp(X_1(t)), \exp(X_2(t)), \exp(Y(t)))\) is a positive solution of system (1.5). Therefore, according to Theorem 2.1, we drew the conclusion that the solution of (2.1) is in \(S^* = \{(X_1, X_2, Y) | \ln m_1 \leq X_i \leq \ln M_1, \ln m_2 \leq Y \leq \ln M_2, i = 1, 2\}\), for all \(t > T\).

and the set \(S^*\) is compact. Hence we need only to show that system (2.1) has a global attractive almost periodic solution.

To do this, we need only to check condition of the system (2.1) satisfies the requirement in Lemma 2.2. So we consider the product system of (2.1)
\[
\begin{align*}
\dot{X}_1 &= a_1(t) - b_1(t) e^{X_1} + \frac{c_1(t) e^{Y}}{\Delta_1(t, e^{X_1}, e^{Y})} + D_1(t) (e^{(X_2 - X_1)} - 1)
\dot{X}_2 &= a_2(t) - b_2(t) e^{X_2} - \frac{c_2(t) e^{Y}}{\Delta_2(t, e^{X_2}, e^{Y})} + D_2(t) (e^{(X_1 - X_2)} - 1)
\dot{Y} &= -d(t) + \frac{f_1(t) e^{X_1}}{\Delta_1(t, e^{X_1}, e^{Y})} + \frac{f_2(t) e^{X_2}}{\Delta_2(t, e^{X_2}, e^{Y})} - q(t) e^{Y} 
\dot{X}_1^* &= a_1(t) - b_1(t) e^{X_1^*} + \frac{c_1(t) e^{Y^*}}{\Delta_1(t, e^{X_1^*}, e^{Y^*})} + D_1(t) (e^{(X_2^* - X_1^*)} - 1)
\dot{X}_2^* &= a_2(t) - b_2(t) e^{X_2^*} - \frac{c_2(t) e^{Y^*}}{\Delta_2(t, e^{X_2^*}, e^{Y^*})} + D_2(t) (e^{(X_1^* - X_2^*)} - 1)
\dot{Y}^* &= -d(t) + \frac{f_1(t) e^{X_1^*}}{\Delta_1(t, e^{X_1^*}, e^{Y^*})} + \frac{f_2(t) e^{X_2^*}}{\Delta_2(t, e^{X_2^*}, e^{Y^*})} - q(t) e^{Y^*} 
\end{align*}
\]

(2.2)

define Lyapunov function as
\[
V(t, Z(t), Z^*(t)) = |X_1(t) - X_1^*(t)| + |X_2(t) - X_2^*(t)| + |Y(t) - Y^*(t)|, \text{ then we obtain that}
\]

(I) \(\text{set } a(|Z(t) - Z^*(t)|) = b(|Z(t) - Z^*(t)|)\)
\[
= |X_1(t) - X_1^*(t)| + |X_2(t) - X_2^*(t)| + |Y(t) - Y^*(t)| \text{ the requirement (I) of Lemma 2.2 is satisfied;}
\]

(II) suppose that \(\hat{Z}(t) = (\hat{X}_1(t), \hat{X}_2(t), \hat{Y}(t)), \hat{Z}^*(t) = (\hat{X}_1^*(t), \hat{X}_2^*(t), \hat{Y}^*(t))\) is an arbitrary solution of product system (2.2), then
\[
\begin{align*}
|V(t, Z(t), Z^*(t)) - V(t, \hat{Z}(t), \hat{Z}^*(t))|
&= \left| \sum_{i=1}^{2} |X_i(t) - X_i^*(t)| + |Y(t) - Y^*(t)| \right|
&\leq \sum_{i=1}^{2} |X_i(t) - X_i^*(t)| + |Y(t) - Y^*(t)|
&\leq \sum_{i=1}^{2} (|X_i(t) - X_i^*(t)| + |X_i^*(t) - \hat{X}_i(t)| + |Y(t) - \hat{Y}(t)|) + |Y^*(t) - \hat{Y}^*(t)|
&= |Z(t) - \hat{Z}(t)| + |Z^*(t) - \hat{Z}^*(t)|
\end{align*}
\]

thus the condition (II) of Lemma 2.2 is satisfied. Now we reach the position to check the requirement (III) in Lemma 2.1 is also satisfied. Based on the discussion above, we know that the solution of (2.2) is in \(S^*\). We assume \(t > T\) in the following for what we will study is global attractive almost periodic solution. For convenience, we still denote \(e^{X_i(t)}, e^{X_i^*}, e^{Y(t)}, e^{Y^*(t)}\) by \(x_i(t), y(t), x_i^*(t), y^*(t)\) respectively, and let \(\delta(t) = \Delta_1(t, \hat{x}_i(t), y(t)), \Delta_i^*(t) = \Delta_1^*(t, \hat{x}_i^*(t), y^*(t)), i = 1, 2\).

Calculating the upper right Dini derivatives of \(V(t)\) along the solution of(2.2), we obtain
\[
D^+ V(t) = \sum_{i=1}^{2} \frac{2}{X_i(t) - X_i^*(t)} (X_i(t) - X_i^*(t)) + \frac{2}{X_i(t) - X_i^*(t)} (X_i(t) - X_i^*(t)) Y(t) - Y^*(t) \]
\[
= \frac{X_1(t) - X_1^*(t)}{|X_1(t) - X_1^*(t)|} \left[ -b_1(t) (x_1(t) - x_1^*(t)) + D_1(t) \frac{x_2(t)}{x_1(t)} - D_1(t) \frac{x_2(t)}{x_1(t)} c_1(t) \frac{1}{\Delta_1(t) \Delta_1^*(t)} \right]
\]

(2.3)
\[c_1(t)\beta_1(t)x_1(t) + D_1(t) \frac{x_2(t)}{x_1(t)x_1^*(t)} + D_2(t) \frac{1}{x_2^*(t)}\]

\[\frac{f_1(t)\gamma_1(t)y(t) + f_1(t)\alpha_1(t)}{D_1(t) \Delta_1^*(t)} + \frac{x_1(t)}{\Delta_1(t) \Delta_1^*(t)} - b_1(t)|x_1(t) - x_1^*(t)|\]

\[+ \frac{c_2(t)\beta_2(t)x_2^*(t)}{\Delta_2(t) \Delta_2^*(t)} + D_2(t) \frac{x_2^*(t)}{x_2^*(t)} + D_1(t) \frac{1}{x_2^*(t)}\]

\[+ \frac{f_2(t)\gamma_2(t)y(t) + f_2(t)\alpha_2(t)}{\Delta_2(t) \Delta_2^*(t)} - b_2(t)|x_2(t) - x_2^*(t)|\]

\[+ \left[\frac{c_1(t)\beta_1(t)x_1(t) + c_1(t)\alpha_1(t) - f_1(t)\gamma_1(t)x_1(t)}{\Delta_1(t) \Delta_1^*(t)}\right]x_1(t) - x_1^*(t)\]

\[+ \frac{c_2(t)\beta_2(t)x_2(t) + c_2(t)\alpha_2(t) - f_2(t)\gamma_2(t)x_2(t)}{\Delta_2(t) \Delta_2^*(t)}\]

\[\frac{q(t)}{\Delta_2(t) \Delta_2^*(t)}|y(t) - y^*(t)|\]

\[\leq \left[\frac{c_1(t)\beta_1(t)M_1 + f_1(t)\gamma_1(t)M_2 + f_1(t)\alpha_1(t)}{\Delta_1(t, m_1, m_2)^2}\right]x_1(t) - x_1^*(t)\]

\[+ \frac{D_1(t) \frac{M_2}{m_1} + D_2(t) \frac{1}{m_1} - b_1(t)}{\Delta_1(t, m_1, m_2)^2} |x_1(t) - x_1^*(t)|\]

\[+ \left[\frac{c_2(t)\beta_2(t)M_1 + f_2(t)\gamma_2(t)M_2 + f_2(t)\alpha_2(t)}{\Delta_2(t, m_1, m_2)^2}\right]x_2(t) - x_2^*(t)\]

\[+ \left[\frac{-q(t) + c_1(t)\alpha_1(t)M_1 + c_1(t)\alpha_1(t)}{\Delta_1(t, m_1, m_2)^2}\right]f_1(t)\gamma_1(t)\frac{x_1(t)}{m_1}\]

\[\Delta_1(t, m_1, m_2) + \frac{c_2(t)\beta_2(t)M_1 + c_2(t)\alpha_2(t)}{\Delta_2(t, m_1, m_2)^2}\]

\[\frac{f_2(t)\gamma_2(t)\frac{x_2(t)}{m_1}}{\Delta_2(t, m_1, m_2) \Delta_2(t, m_1, m_2)}|y(t) - y^*(t)|\]

\[\leq -\frac{C}{2}(x_1^*(t) - x_1^*(t))^2 + |x_2(t) - x_2^*(t)| + |y(t) - y^*(t)|\]

\[= -CV(t).\]

Therefore, the requirement (III) of Lemma 2.2 is also satisfied. Then we conclude that system (2.2) has an unique almost periodic solution which is in set \(S^*\) and globally attractive. Hence the system (2.1) has an unique almost periodic solution which is in set \(S\) and global attraction. The proof is complete.

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