REPRESENTING SONAR/RADAR RETURNS AS A MARKOV PROCESS

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ABSTRACT

There is a significant advantage in detecting signals sequentially. Generally, sequential detection minimizes the time it takes to make a decision on the average. This is an important requirement in sonar and radar applications. Operationally, sequential analysis acts much like a human operator in that it defers a decision until a high confidence level in the signal is achieved. Whereas, noise is quickly rejected.

In the paper we formulate the multiple transmission application in terms of a sequential detection problem based on the likelihood ratio. The first application is for a Rayleigh fluctuating target. The likelihood ratio is constructed based on probability density functions. This gives the optimum receiver for the assumed conditions. A more general model based on a Markov process is considered next. In this case the current return is correlated with past returns. It is shown that correlated returns improve detection performance. Extending this model further, correlated returns with non-Gaussian statistics are formulated next using mixture densities.

INTRODUCTION

Successive returns from sonar or radar transmissions are optimally processed sequentially based on the likelihood ratio \([l]\). The optimum property of sequential analysis ensures that the time it takes to make a decision on the average is minimized. This is, of course, the main reason behind the many applications of Wald's sequential detection theory. However, from a practical point of view, the sequential procedure is a natural way to detect a signal. For example, consider an operator viewing a display. The operator waits until he is sure that a signal is present. But, this is the basic philosophy behind sequential detection. A signal is detected only when there's sufficient cause. On the other hand, noise is rejected by the sequential procedure.

For our application the sequential procedure is applied at a matched filter output. The matched filter is based on the active transmission, which may be CW, LFM, or coded pulse. The receiver for successive returns is based on the likelihood ratio. Therefore, for each return the loglikelihood ratio \([T_n]\) is compared with two thresholds, \(T_nA\) and \(T_nB\). If \(T_n \geq T_nA\) a signal is present whereas, if \(T_n \leq T_nB\) noise is present. However, if \(T_nB < T_n < T_nA\), a decision is deferred until the next return is received. This procedure minimizes the average detection time. However, the amount of data accumulated is now a random variable. In order to prevent long decision times from occasionally occurring a non-constant decision boundary is sometimes employed. This is called a truncated or forced decision test, because at some time a decision will be forced to occur. It has been shown that if a truncated test is used the decision time can be reduced further, but at the expense of higher rates.

Wald was able to relate thresholds in terms of false alarm \((\alpha)\) and false dismissal \((\beta)\) probabilities by neglecting the excess over the boundaries. This means that the sequential detection procedure terminates on the boundaries. This is a good assumption for small to moderate signal-to-noise ratios. However, for high signal-to-noise ratio problems more exact methods are needed.

The performance of sequential detection procedures are usually based on the operating characteristic function (OCF) and the average sample number (ASN) as originally defined by Wald.

The OCF is defined as follows:

\[
L(h) = \frac{(A^h - 1)}{(A^h - B^h)},
\]

where \(L(h)\) is the probability of accepting noise \((H_0)\) as a function of \(h\). Whereas, the probability of accepting the signal \((H_1)\) is, \(1 - L(h)\), since it is assumed that the test will eventually terminate. The parameter \(h\) is a function of signal parameters. It can be obtained from the moment generating function

\[
E[e^{T_nh}] = 1.
\]

Notice that the equation is satisfied at \(h = 1, h = -1\), and \(h = 0\). However, the objective is to find an analytic expression for \(h\) which satisfies the equation for \(-1 \leq h \leq 1\).
But it is not always possible to find a closed form solution. For these cases, a small signal-to-noise ratio solution is often useful.

The other performance measure, the ASN, is defined as follows.

\[
\text{ASN} = \tilde{n} = \frac{L(b) \ln B + (1 - L(b)) \ln A}{E[T_i]}, \quad h \neq 0
\]

\[
\tilde{n} = \frac{L(0) (\ln B)^2 + (1 - L(0)) (\ln A)^2}{E[T_i]}, \quad h = 0.
\]

Several examples will now be considered. The first example is for a Rayleigh fluctuating signal. Here the signal is assumed to be stationary for the pulse duration. But successive returns fluctuate with Rayleigh statistics. In the Rayleigh model the returns are assumed statistically independent. However, if the returns are correlated detection performance can be improved over the Rayleigh case. We incorporate correlated returns by assuming the returns obey a Markov process and then employ this assumption in the likelihood ratio receiver [2].

Since the returns need not follow Gaussian statistics, our model is generalized further by developing a likelihood ratio receiver for a mixture density [3].

RAYLEIGH FLUCTUATING SIGNAL

Before the likelihood ratio is applied the data are passed through a matched filter. Therefore, in general, for each return there is an inphase component and a quadrature component to construct the likelihood ratio. The Rayleigh density is based on both components which are assumed Gaussian. For convenience and without loss of generality we shall only consider one component, say the inphase component, for the other densities.

The loglikelihood ratio has the following form

\[
\text{LLR} = \sum_{i=1}^{n} T_i,
\]

where

\[
T_i = \ln \left( \frac{f_i(x_i)}{f_0(x_i)} \right)
\]

and \(f_i(x_i), j = 0,1\) represents the Rayleigh density under noise (\(j = 0\)) and signal and noise (\(j = 1\)).

Specifically,

\[
f_i(x_i) = \frac{x_i}{1 + S_1} \exp\left(-\frac{x_i^2}{2(1 + S_1)}\right) u(x_i)
\]

\[
f_0(x_i) = x_i \exp\left(-\frac{x_i^2}{2}\right) u(x_i)
\]

where \(S_1\) is the designed single return signal-to-noise ratio.

Further,

\[
T_i = -\ln(1 + S_1) + \left(\frac{S_1}{1 + S_1}\right) x_i^2 / 2
\]

represents the receiver structure. It uses the energy of the return pulse at the match filter output. Notice the first term is included. This is required in a sequential test.

From the definition of the ASN \(E[T_i]\) is needed. This follows as,

\[
E[T_i] = -\ln(1 + S_1) + S_1 \left(\frac{1 + S}{1 + S_1}\right)
\]

and for the weak signal case, it reduces to

\[
E[T_i] = -\frac{S^2}{2} + SS_1.
\]

Here \(S\) is the actual signal-to-noise ratio.

Example 1:

Let, \(a = 1 nA = 18.3, b = 1 nB = -2.3\)

\(a = 10^{-8}, B = 10^{-1}\)

Then,

\[
\text{ASN} = \tilde{n} = \frac{L(b)(1 - L(b))a}{E[T_i]},
\]

where, \(L(1) = 1 - a, \) and \(L(-1) = B\).

<table>
<thead>
<tr>
<th>SNR</th>
<th>(\tilde{n}_0)</th>
<th>(\tilde{n}_1)</th>
</tr>
</thead>
<tbody>
<tr>
<td>16 dB</td>
<td>.84</td>
<td>.5</td>
</tr>
<tr>
<td>10 dB</td>
<td>1.54</td>
<td>2.14</td>
</tr>
<tr>
<td>6 dB</td>
<td>2.84</td>
<td>6.8</td>
</tr>
<tr>
<td>3 dB</td>
<td>5.3</td>
<td>18</td>
</tr>
</tbody>
</table>

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Several cases are shown. For the high SNR case (SI = 16 dB), Wald's formulation does not give an accurate prediction. For the Gaussian case we shall employ an exact method. For lower SNR's it is very accurate. Notice that as SNR decreases more returns are needed to detect the signal under the prescribed error rates. On the other hand, a 3 dB increase in SNR decreases the number of returns substantially.

The operating characteristic function is obtained from the following equation

\[ \int_0^e \frac{x^2}{1+S_1} e^{2(1+S_1)} dx_1 = 1 \]

This reduces to the nonlinear equation,

\[ \left( \frac{1}{1+S_1} \right)^h = 1 - \frac{S_1(1+S_1)}{1+S_1} h \]

For the weak signal approximation we obtain,

\[ h = 1 - 25/S_1 \]

Example 2: For the same parameters given in equation 1, the OCF, \( L(h) = \frac{(e^{ah} - 1)}{(e^{ah} - e^{bh})} \), has the following values:

<table>
<thead>
<tr>
<th>h</th>
<th>1.25</th>
<th>0.1</th>
<th>0</th>
<th>-1</th>
<th>-1</th>
</tr>
</thead>
<tbody>
<tr>
<td>L(h)</td>
<td>0.995</td>
<td>0.9</td>
<td>0.888</td>
<td>0.765</td>
<td>0</td>
</tr>
</tbody>
</table>

ONE-DIMENSIONAL GAUSSIAN

To motivate the Markov process discussed next a one-dimensional Gaussian density will be considered first. Figure 1 shows the configuration. For \( H_1: f_1 \sim N(\theta_1/2,1) \), and for \( H_0: f_0 \sim N(- \theta_1/2,1) \). The loglikelihood ratio reduces to

\[ T_n = \theta_1 \sum_{i=1}^N x_i \]

where \( \theta_1 \) represents the design SNR for this case. If \( H_1 \) is true, then \( T_n \) sums the returns until the upper threshold is crossed. Whereas, if \( H_0 \) is true the lower threshold will eventually be crossed.

To show the performance of the exact method the data are partitioned [2] into two regions at \( x_1 = 0 \). The partitioned regions are then assigned scores based on the loglikelihood ratio. The results are shown in figure 2 for several SNR's given a design SNR of 16 dB.

MARKOV PROCESS

Figure 3 gives the configuration for the two-dimensional Gaussian case. Here, the current return, \( x_1 \), at the matched filter output is correlated with the previous return, \( x_{1-1} \). This is called a Markov process. A generalization of this basic idea is given in reference 2. The use of a Markov process representation for radar is given in reference 4. The approach presented here is more general than reference 4 because, many partition regions can be taken into account, non-Gaussian densities can be treated, and the problem is formulated in terms of the sequential detection procedure.

A simple proof can be given to show that correlation will improve the probability of detection. Let the densities under \( H_0 \) and \( H_1 \) have zero means and unit variances. If the correlation was also zero then it would be impossible to distinguish between \( H_0 \) and \( H_1 \). But if the signal return was correlated with previous returns this would be enough to decide in favor of \( H_1 \).

Figure 4 gives the performance for the Markov process using the exact method. Here, instead of two partition regions, there are four partition quadrants. Each are assigned scores based on the loglikelihood ratio. As far as the figure is concerned, correlation increases detection compared with figure 2. Inspection of the loglikelihood ratio, on the other hand, reveals that correlation shifts its score structure to favor detection when correlated data are present. Whereas, uncorrelated noise is quickly rejected.

NON-GAUSSIAN STATISTICS

In order to represent correlated returns with non-Gaussian statistics the probability density function is expressed as a mixture of two Gaussian probability density functions [3]. The mixture density has the form

\[ f(x,y) = (1 - \lambda) f_1(x,y) + \lambda f_2(x,y) \]

where the marginal densities obey the equation, \( f_1(x) = f_2(x) \). Therefore, \( \sigma_1 = \sigma_2 = \sigma \).

The parameter \( \lambda \) can be thought of as a duty cycle for each mixture density.

For this model, with \( \sigma = 1 \), the correlation coefficient \( \rho \) has the following form.
We can test to see if \( \rho \) is from a Gaussian or a mixture non-Gaussian probability density by considering a forth order estimate. For example, for a purely Gaussian probability density,

\[
E[x^2y^2] = 1 + 2\rho^2
\]

Whereas, for a mixture non-Gaussian probability density

\[
E[x^2y^2] = 1 + 2(1 - \lambda) \rho_1^2 + \lambda \rho_2^2
\]

Notice that, \( \rho^2 = (1 - \lambda) \rho_1^2 + \lambda \rho_2^2 \)

unless \( \rho_1 = \rho_2 \) or \( \lambda = 0, 1 \). All other cases are non-Gaussian.

CONCLUSION

Sequential analysis minimizes the time to make a decision on the average. This is important in sonar and radar applications. Therefore, the performance of a multiple transmission system was formulated in terms of sequential detection procedures. Uncorrelated returns with Rayleigh statistics was analyzed. Then this model was generalized to handle correlated returns. It was shown that correlation improves the probability of detection if the data were modeled as a Markov process. In addition, a mixture density was introduced to predict system performance of correlated returns with non-Gaussian statistics. A method was given using a fourth order estimate to test between correlated returns with Gaussian statistics and correlated returns with non-Gaussian statistics.

REFERENCES


