COMPARISON OF THE ZIV-ZAKAI LOWER BOUND ON TIME DELAY ESTIMATION WITH CORRELATOR PERFORMANCE

John P. Ianniello
Naval Underwater Systems Center, New London, CT 06320

Ehud Weinstein
Woods Hole Oceanographic Institution, Woods Hole, MA 02543

Anthony Weiss
School of Engineering, Tel-Aviv University, Tel-Aviv, Israel 69978

ABSTRACT

The Ziv-Zakai Lower Bound (ZZLB) on the mean square error (m.s.e.) of time delay estimators is compared with theoretical and computer simulation results for time delay estimation via cross-correlation. Comparisons are made for both lowpass and narrowband signal spectra. For both signal spectra it is shown that for sufficiently large time-bandwidth product the correlator performance is very close to the ZZLB in both the small and large error regions. This establishes the cross-correlator as an optimal instrumentation (in the m.s.e. sense) while demonstrating that the ZZLB is an extremely tight lower bound. The ZZLB is further used to accurately predict the threshold signal-to-noise ratio (SNR) above which the cross-correlator performance is closely characterized by the Cramer-Rao Lower Bound (CRLB).

INTRODUCTION

Time delay estimation between signals radiated from a common point source and observed at two spatially separated sensors, each of which also receives uncorrelated noise, is a problem of considerable practical interest in underwater acoustics. A common method for estimating the differential time delay is to cross-correlate the two sensor outputs, average for T seconds, and select as the estimate the delay associated with the peak of the correlogram. It is also well known that the cross-correlator, with proper filtering, is the Maximum Likelihood (ML) estimator of the sensor-to-sensor delay. Approximate theory and extensive computer simulation of the cross-correlator m.s.e. performance are reported in [1], where it is shown how the threshold SNR exists such that for SNRs greater than threshold the resulting m.s.e. approaches the absolute minimum characterized by the CRLB. For SNRs below threshold the cross-correlator performance rapidly deteriorates due to large anomalous or ambiguous estimates.

While the anomaly and ambiguity effects are fundamental and unavoidable features of the delay estimation problem, independent of the signal processing technique, it is still unclear whether the cross-correlation estimator is optimal for SNR’s below threshold. Thus, it is important to compare the cross-correlator performance with some useful lower bound for the low SNR mode of operation.

A first step in this direction is reported in [2] where a simplified version of the Barankin Bound (BB), derived in [3], is compared with the correlator performance for narrowband signals. It is shown there that the threshold SNR achieved by the correlator is significantly larger than that predicted by the BB.

Another bounding technique based on a modified (improved) version of the ZZLB is developed in [4]. A direct comparison between the simplified version of the BB and the improved version of the ZZLB is carried out in [4] where it is indicated that the ZZLB is a greater (thus tighter) lower bound than the simplified BB, at least for narrowband signals. The important contribution of this paper is the direct comparison of the ZZLB and the correlator for both lowpass and narrowband signal spectra, thus allowing a joint assessment of how tight the ZZLB is and of how nearly optimal the cross-correlator is.

PROBLEM DESCRIPTION: ANOMALIES AND AMBIGUITIES

We consider two sensors one of which receives a random signal \( S(t) \) plus uncorrelated noise \( N_1(t) \), and the other of which receives a time delayed version of the signal \( S(t - D) \) plus uncorrelated noise \( N_2(t) \). Both sensor outputs are observed during a common time period of T seconds. An estimate \( \hat{D} \) of the time delay is to be formed. An a priori search region \( \pm D_0/2 \) for \( D \) is known; this comes about, perhaps, from the known sensor separation and the known sound velocity. To obtain some feeling for the source of the anomalous estimates assume that the estimate, \( \hat{D} \), is formed by selecting the peak value of the cross-correlation estimate of the two sensor outputs. Figure 1 shows the correlator output \( Z(\hat{D}) \) for a lowpass signal spectrum. Assume the
oscillatory correlation peaks within ±Tc/2 can exceed the peak value at D = 0. These errors
are referred to as ambiguous estimates; they may still yield some useful information about the
time delay, however.

In the next section we shall compare the cross-correlator performance above and below the
threshold with the ZZLB. The section is divided into two sub-sections: first, we consider
lowpass signals (and the associated anomaly effect); second, we consider narrowband signals
(and the associated ambiguity effect).

**COMPARISON OF THE ZZLB WITH CORRELATOR PERFORMANCE**

**Lowpass Signals**

In this sub-section we consider signals whose power spectra is distributed about zero frequency. For computational convenience we shall assume that both signal and noise are spectrally flat within |f| ≤ B/2 Hz and zero outside.

In Figure 3 we show the mean square estimation error, normalized by D²/12 (the mean square value when the error is uniformly distributed in ±D/2), as a function of the system SNR, defined by

\[ \text{SNR} = \frac{(S/N)^2}{1 + 2S/N} \]  

where S/N is the SNR at the individual sensors. The plot is for BT = 128 and BD₀ = 16. The

Figure 1. Correlator Output for Lowpass Signals

true value of D is zero. Z(0) for high SNR is shown in the top panel. Here the errors are small and the correlation peak occurs close to the true value, say within a correlation time *Tc/2. This is the situation that must apply for the CRLB to be reachable. As SNR decreases the peak near D = 0 exhibits larger and larger variance until suddenly, at a certain SNR, large peaks occur far from the true value. These peaks will be uniformly spread throughout the a priori interval and at this point the estimates are useless. These may be referred to as anomalous estimates.

Figure 2 shows the correlator output for a narrowband signal spectrum. Now the envelope of Z(0) is filled with a cosine carrier. The CRLB applies only when the peak value occurs within plus or minus one half cycle of the true value. At low SNR two types of large errors can occur. As in the lowpass case, anomalous estimates well removed from the true value can occur. Further, a second type of large error can now occur, as indicated, even if only the smaller region ±Tc/2 is searched, since one of the highly

Figure 2. Correlator Output for Narrowband signals

Figure 3. Normalized MSE vs. System SNR - Lowpass Signal
lowest solid curve is the CRLB while the upper solid curve is the ZZLB calculated from the exact results of [4]. The dashed line is an approximate theoretical result for the variance of the peak selecting correlator for large errors, calculated as described in [1].

As described in [1] and [2] a computer simulation was run to verify the theoretical results. Cross-correlation was performed via FFT processing. Data were acquired in both a gated and an ungated mode. In the gated mode the correlation peak closest to the known true delay that also fell within a correlation distance of the true value was used to calculate the variance. This restricts the variance calculation to local errors only and allows a comparison with the CRLB. These data are shown as the x's in Figure 3; the bars represent 95% confidence intervals. The agreement with the CRLB at high SNR is good, as expected; these data serve mainly as a check on the quality of the simulation. These data are shown as the circles in Figure 3. Note the close agreement between the ungated mode data and the approximate theory shown by the dashed line.

The most important observation to be made from Figure 3 is the close agreement between the correlator and the ZZLB. Indeed in the threshold region the two curves are displaced by roughly 1 dB in input S/N at the sensors. This establishes the ZZLB as a very tight bound and at the same time shows that the cross-correlator is very nearly an optimal instrumentation in the minimum m.s.e. sense.

The threshold SNR, denoted by SNR_{Th}, above which an essentially anomaly-free estimate can be achieved is of theoretical and practical importance. An extensive analysis of the ZZLB for lowpass spectra is carried out in [5] where it is shown that SNR_{Th} for the lowpass case is given to an excellent approximation by

$$ SNR_{Th} = \frac{2}{BT} \left( F_{+}^{-1}( \left[ \frac{3}{BD_0} \right]^2 ) \right)^2 $$

Here $F(x) = \frac{x^2}{2} \Phi(x)$ where $\Phi(x)$ is the integral from $x$ to $-\infty$ of the Gaussian probability density function. $F_{+}^{-1}(x)$ denotes the larger of the two solutions of $F^{-1}(x)$. SNR_{Th} is an approximation to the SNR at which the ZZLB is 3 dB above the CRLB. Thus for BT = 128 and BD_0 = 16, SNR_{Th} = -7.4 dB which is within 1 dB of the threshold point of the cross-correlator as observed from Figure 3.

Comparisons have also been made for other BT products. For BT equal 1280 the correlator performance is again within 1 dB of the ZZLB near threshold. For BT equal 64 the correlator is within 2 dB of the ZZLB. For BT equal 32 and 16 the correlator performance is 5 to 10 dB to the right of the ZZLB near threshold. Thus, at present we assert that, for lowpass signals, the ZZLB is tight and the correlator is nearly optimal for BT greater than, say, 50. For BT less than 50 we can only say, at present, that either the correlator is not optimal and/or the ZZLB is not tight.

Narrowband Signals

We now consider signals whose power spectra are narrowly distributed about some center frequency $f_0$. For computational convenience we shall assume that both signal and noise are spectrally flat within $|f - f_0| < B/2$ and zero elsewhere. In Figure 5 we show the mean square estimation error, normalized by $1/f_0$, as a function of system SNR. The plot is for $f_0/B = 10$, $BT = 16$ and $BD_0 = 2$. The lowest solid curve is the CRLB; the next solid curve is the simplified BB calculated from [3] and included here for reference; the upper solid curve is the ZZLB calculated from the exact results of [4]. The dashed line is an approximate theoretical result for the variance of the peak selecting correlator. This was found by an improved version of the method given in [2] and is based on the calculation of the probability that either of the two correlation peaks immediately adjacent to the peak at the true value of time delay are greater than the true peak. Unlike the method employed in [2] this technique does not assume that the two adjacent peaks are uncorrelated. The calculation is only valid for large SNR.

**Figure 4. Normalized MSE vs. System SNR**

ICASSP 83, BOSTON
A computer simulation was run as described above. The gated mode data were again within the 95% confidence limits of the CRLB for high SNR; these have not been included in Figure 4 for simplicity. The ungated data are shown by the circles. They agree quite well with the theoretical result for the correlator but the correlator variance is much larger than the ZZLB at high SNR. In fact, for BT = 16, the correlator, as implemented, will continue to make occasional large errors even at infinitely high SNR. This occurs because the correlator includes new data in the estimate at each lag value; as a result it cannot be guaranteed that, even at infinite SNR, the peak value will be at the true value of time delay.

Despite this, one feels that at least at infinite SNR some processing scheme which makes no errors ought to be realizable. One way to ensure that no errors are made at infinite SNR is as follows. For infinite SNR and for a true time delay of zero, the correlator output will always be symmetrical about the true value of time delay. Thus, if one checks the symmetry of the three largest peaks one can always choose the correct peak. We employed this symmetry checking technique at non-infinite SNR to see if some improvement still resulted. This scheme is not practical but it does give some insight into the best one may be able to achieve. The data for the symmetry checker are shown by the x's in Figure 4. These data are quite a bit closer to the ZZLB than those for the simple peak selecting correlator. Since these data come from a realizable processor they also establish an upper bound; since they are within 2-3 dB in system SNR of the ZZLB they show that the ZZLB is at least this tight for small BT.

The situation is considerably simpler for larger BT products. In Figure 5 we show theoretical results for the normalized variance for BT = 1000, fL/B = 10 and BD = 2 as before. Now we see that there is a threshold SNR for the correlator is within 1 dB of the ZZLB. Thus, for narrowband signals and large BT the ZZLB is shown to be very tight. Again we are interested in the threshold SNR above which an essentially ambiguity-free estimate of the sensor-to-sensor delay can be obtained. An extensive analysis of the ZZLB for narrowband signals is carried out in [4] where it is shown that for narrowband signals the threshold point is given to an excellent approximation by

$$\text{SNR}_{Th} = \frac{6}{\pi^2} BT \left[ \frac{1}{24} \left( \frac{8}{f_0} \right)^2 \right] \left( \frac{\beta}{B} \right)^2$$

(3)

where as in Eq. (2), SNR$_{Th}$ is defined as the point at which the ZZLB is 3 dB above the CRLB. Thus for BT = 1000 and fL/B = 10, SNR$_{Th}$ = -1.7 dB which is within 1 dB of the threshold point of the cross-correlator, as observed from Figure 5.

By comparing actual correlator performance with the ZZLB we have shown that the ZZLB is indeed a tight bound. For lowpass signals and BT greater than 50 the correlator threshold SNR is within 1 dB of the ZZLB. For narrowband signals and large BT the correlator threshold again is within 1 dB of the ZZLB; for small BT (BT = 16) the correlator (as implemented here) does not reach the ZZLB but an alternate scheme, based on a symmetry check, comes within 2-3 dB of the ZZLB near threshold.

REFERENCES


