Information Theoretic Optimal Broadband Matching for Communication Systems

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Abstract—We present an information theoretic perspective on optimal broadband matching. Most of the broadband matching and communication theory literature assumes a frequency-flat reflection coefficient. We derive the optimal capacity and matching network characteristic for a broadband system from an information theoretic view point while incorporating the broadband matching limitations. We also propose an iterative algorithm to arrive at the optimal solution. We illustrate by an example that a white characteristic for a matching network is often sub-optimal for a system with a non-white transfer function.

I. INTRODUCTION

During the last decade, multiple-input-multiple-output (MIMO) systems have been shown to be a promising way to achieve high data rates [1] and alleviate the problem of signal fading arising from multipath propagation in wireless channels [2]. However, most of these advantages can only be realized for large inter-element spacings where as, the current demand for high data rate systems is fueled by handheld and portable devices. These devices are physically constrained to support a large array of elements due to the performance degradation resulting from signal correlation and mutual coupling among closely spaced elements. Therefore, modeling and performance evaluation of such coupled arrays has received a lot of attention lately [3]-[4]. Multiport conjugate matching networks have been shown to completely compensate for this loss even at very close inter-element spacings, however, these networks suffer from severe bandwidth limitations [5]. Hence, in evaluating the capacity limits of compact MIMO systems, it is imperative to undertake an information theoretic approach to understand the role that broadband matching plays on the performance of coupled arrays.

A recent study [6] has analyzed bounds on the spectral efficiency of an arbitrary antenna array inserted in a sphere, under Rayleigh fading conditions. Each array-element is represented via a resonance model and a corresponding Q-factor while the fundamental limits from the realms of antenna theory and broadband matching are used to get an estimate of the maximum permissible bandwidth of the system. It however, assumes a frequency-flat reflection coefficient for the matching network and no knowledge of the channel at the transmitter. The results are presented in the form of an upper bound on the spectral efficiency.

In practice, matching networks are traditionally designed assuming a frequency-flat characteristic for a specified bandwidth, and then realized using a well-known filter response such as Butterworth, Tchebyshev, etc. In this work, we seek to analyze the optimal end-to-end design of a broadband communication system from an information theoretic perspective while addressing the bandwidth limitations arising from broadband matching theory. A multiport extension to the broadband matching theory for coupled ports is not available, however, by considering a broadband single transmit-receive pair, we hope to gather key insights into the more general MIMO model.

The model considered in this paper, applies to any broadband communication system, wired or wireless. Narrowband systems can be treated as a special case. The optimal design is broken into two mutually dependent parts: an optimal power spectrum employed at the transmitter and an optimal matching network applied at the receiver. The results from a simple numerical example suggest that the bandwidth of a broadband transceiver has an optimal value and that optimal matching networks need not have a frequency-flat characteristic.

II. SYSTEM MODEL

Consider a wireless communication system with two antennas placed a distance $d$ apart communicating over the line-of-sight (LOS) at carrier frequency $f_0$. From Friis’ transmission equation [7], the received power-spectral-density (PSD) is given by

$$P_r(f) = P_t(f)G_t(\theta_t, \phi_t)G_r(\theta_r, \phi_r) \left(\frac{c}{4\pi fd}\right)^2$$

(1)

where, $P_t(f)$ is the transmitted PSD, $G_t, G_r$ model the antenna gains in the cylindrical co-ordinates and $c$ is the speed of light. For simplicity, we assume both the antennas to be omni-directional with unit gain, i.e., $G_t = G_r = 1$.

The Shannon capacity [8] of such a communication system is given by

$$C = \max_{P_t(f)} \int_{f \in W} \log_2 \left(1 + \frac{|H(f)|^2 P_t(f)}{N(f)}\right) df$$

(2)
where \( N(f) \) represents the noise PSD at the receiver, and \( |H(f)|^2 \) represents the overall transfer function of the system

\[
|H(f)|^2 = \left( \frac{c}{4\pi f d} \right)^2
\]

The notation, \( f \in W \), represents the frequency range such that \( P_t(f) \neq 0 \). For ease of mathematical tractability, we shall use natural logarithm to use the following definition of capacity in the rest of the paper

\[
C = \max_{P_t(f)} \int_{f \in W} \ln \left( 1 + \frac{|H(f)|^2 P_t(f)}{N(f)} \right) df
\]  

(3)

If the total power available at the transmitter is constrained to be \( P_0 \), i.e.,

\[
\int_{f \in W} P_t(f) df \leq P_0
\]  

(4)

then for such a system, the optimal power allocation is the well-known water-pouring algorithm [9] given by

\[
P_t(f) = \left[ \frac{1}{\lambda_{wp}} - \frac{N(f)}{|H(f)|^2} \right]^+\]

(5)

where, \([x]^+ = \max(0, x)\) and \( \lambda_{wp} \) is chosen to satisfy (4). This power constraint at the transmitter often results from the maximum power available at the output of the power amplifiers in the transmit RF chain. Ideally, this is the power radiated from the antenna terminals, however, microwave circuits require an impedance matching circuit between the amplifier and the antenna to maximize the power transfer. This brings us to the topic of broadband matching.

III. BROADBAND MATCHING THEORY

In the design of any communication system, it is always desirable to deliver maximum power from a generator (or a source) to a load. This is achieved by embedding a matching network between the generator and the load so as to maximize the power transfer or the signal-to-noise ratio (SNR) (see Fig. 1). This procedure is referred to as impedance matching or simply matching.

The efficiency of a matching network is often characterized by the fraction of the maximum available power from the source \( P_{\text{max,in}}(f) \) that is delivered to the load \( P_{\text{out}}(f) \), [10]

\[
P_{\text{out}}(f) = (1 - |\Gamma(f)|^2) P_{\text{max,in}}(f),
\]

\[
|T(f)|^2 = 1 - |\Gamma(f)|^2
\]

where, \( 0 \leq \Gamma(f), T(f) \leq 1 \) is called the reflection and transmission coefficient, respectively. The reflection coefficient is defined when the network is terminated on both sides in 1-ohm resistances:

\[
\Gamma(f) = \frac{Z_M(f) - 1}{Z_M(f) + 1}
\]

where \( Z_M(f) \) is the driving-point impedance seen looking into the matching network, as shown in Fig. 1.

Friis’ modified transmission equation [11], accounts for this matching efficiency at the transmitter and the receiver by

\[
P_r(f) = P_t(f) |H(f)|^2 (1 - |\Gamma_t(f)|^2) (1 - |\Gamma_r(f)|^2)
\]

where, \( \Gamma_t(f), \Gamma_r(f) \) are the reflection coefficients at the transmit and receive antennas, respectively, the ideal choice for which is \( \Gamma_t(f) = \Gamma_r(f) = 1 \) for \( f \in W \). However, broadband matching theory reveals why this can not be the case.

Bode’s [12, Sec. 16.3] investigation of the bandwidth limitations of a matching network terminated into a resistance \( R \) shunted by a capacitance \( C \), i.e., \( Z_L(s) = (1/R + Cs)^{-1} \), points out the following integral constraint on the reflection coefficient

\[
\int_0^\infty 2\pi \ln \frac{1}{|\Gamma(f)|} df \leq \pi \frac{W}{RC}
\]  

(6)

where \( s = \sigma + j\omega \) is the Laplace variable and \( \omega = 2\pi f \) is the radian frequency. The above equation lays down the trade-off between the magnitude of the reflection coefficient and the bandwidth. In microwave literature, it is customary to assume \( \Gamma(f) = \Gamma_{\text{min}} \) over a bandwidth \( W \) and unity outside, such that from (6)

\[
|\Gamma|_{\text{min}} = e^{-1/2WRC}
\]  

(7)

It is the resulting transducer-power gain characteristic \( G_p(f^2) = 1 - |\Gamma(f)|^2 \), which is realized in practice in the form of a well-known filter response such as Butterworth, Tchebyshev, etc.

Fano [13] further extended these results and laid down the fundamental limits on the design of broadband matching networks for systems with a resistive source and an arbitrary load impedance. For advances in broadband matching theory and its implementation aspects, see [10].

IV. MATCHING CONSTRAINTS

Consider the 2-port network shown in Fig. 1, where a resistive source is connected to a frequency-dependent load \( Z_L(f) \) via a matching network \( N \). According to Darlington [14], any impedance physically realizable using a finite number of linear passive elements, can be considered as the input impedance of a reactive two-terminal-pair network terminated in a pure resistance which can be transformed into 1-ohm by an appropriate ideal transformer in the reactive network (cf. Fig. 2).
As in [13], we turn the network of Fig. 2 end to end, as indicated in Fig. 3, and call the network resulting from the Darlington representation of the load impedance as \( N' \) and the matching network as \( N'' \). The reflection and transmission coefficients for the primed networks are denoted by the corresponding primed quantities with the networks terminated on both sides in 1-ohm resistances. The reflection coefficients \( \Gamma_1(f) \) and \( \Gamma_2(f) \) refer to the whole network \( N \) terminated on both sides in 1-ohm resistances. Since \( N \) is lossless, \( |\Gamma_1(f)| = |\Gamma_2(f)| \).

Using
\[
\Gamma_1(f) = \Gamma'_1(f) + \Gamma''_1(f) - \frac{T'^2(f)}{1 - T^2(f)}\Gamma'_1(f),
\]
it can be shown that [13], the first 2\( K \) coefficients of the Taylor series for \( \ln(1/\Gamma_1(f)) \) about a zero of transmission \( f_z \) of the network \( N' \) (i.e., \( T'(f_z) = 0 \)) of multiplicity \( K \) in the right hand s-plane (RHS) or the imaginary axis will be independent of \( N'' \). And based on the number of such zeros, there are one or more integral equation constraints that need to be satisfied for a physically realizable matching network.

For a load that has \((k+1)\) zeros of transmission at infinity (i.e., \( \lim_{f \to \infty} T^{(k+1)}(f_z) = 0 \)), the Taylor series expansion of \( \ln(1/\Gamma_1(f)) \) about infinity yields an integral equation constraint of the form
\[
\int_0^\infty \omega^{2k} \ln \left[ \frac{1}{\Gamma_1(\omega)} \right] d\omega \leq (-1)^k \frac{\pi}{2} F^{\infty}_{2k+1}
\]
where,
\[
F^{\infty}_{2k+1} = -\frac{1}{2k+1} \sum_i \left( \lambda_{oi}^{2k+1} - \lambda_{pi}^{2k+1} \right) - \frac{2}{2k+1} \sum_i \lambda_{ri}^{2k+1}.
\]
\( \lambda_{oi} \) and \( \lambda_{pi} \) are the poles and zeros of \( \Gamma_1(s) \), respectively, and \( \lambda_{ri} \) are those zeros that lie in the RHS.

For a load that has \((k+1)\) zeros of transmission at zero (i.e., \( T^{(k+1)}(0) = 0 \)), the Taylor series expansion of \( \ln(1/\Gamma_1) \) about zero yields the following constraint
\[
\int_0^{\infty} \omega^{-(2k+1)} \ln \left[ \frac{1}{\Gamma_1(\omega)} \right] d\omega \leq (-1)^k \frac{\pi}{2} F^0_{2k+1}
\]
where,
\[
F^0_{2k+1} = -\frac{1}{2k+1} \sum_i \left( \lambda_{oi}^{-(2k+1)} - \lambda_{pi}^{-(2k+1)} \right) - \frac{2}{2k+1} \sum_i \lambda_{ri}^{-(2k+1)}.
\]

In our study, we consider the case, where the load impedance imposes at least one zero of transmission at infinity:
\[
2 \int_0^\infty \ln \left[ \frac{1}{|\Gamma_1(f)|^2} \right] df \leq F^{\infty}_1
\]

V. OPTIMAL CAPACITY

Consider the system shown in Fig. 4. The Shannon capacity for this system is given by
\[
C = \max_{P, \, F} \int_{f \in W} \ln \left( 1 + |H(f)|^2 \right) \frac{P(f)}{N(f)} df
\]
where, \( \Gamma(f) = \Gamma_1(f), \, P(f) = P_1(f) \) and
\[
|H(f)|^2 = \left( \frac{c}{4\pi fd} \right)^2 \left( 1 - |\Gamma_1(f)|^2 \right).
\]

Observe that we have assumed a fixed or rather a sub-optimal \( \Gamma_1(f) \), optimal only at the center frequency. It is straightforward to include \( \Gamma_1(f) \) in the optimization, however, to illustrate our point, we limit the optimization over \( P(f) \) and \( \Gamma_1(f) \). Also, \( H(f) \) can represent any of the frequency-selective transfer functions, e.g., a wireless fading-channel characteristic or a Digital Subscriber Line (DSL) with appropriate modeling of the impedances. We limit the scope of this paper to LOS communications.

Clearly, (7) is sub-optimal for (13) and lacks any insight into the optimal bandwidth \( W = W_0. \) To begin, let
\[
G(f) = \ln \left( \frac{1}{|\Gamma_1(f)|^2} \right)
\]
such that (12) becomes
\[
\int_{f \in W} G(f) df \leq G_0
\]
where, \( 0 \leq G(f) \leq G_0 \) and \( G_0 = F^\infty_1/2 \).

A. Problem Definition

The aim therefore, is to find the optimal \( P(f) \) and \( G(f) \) that maximizes
\[
C = \max_{P, \, G} \int_{f \in W} \ln \left( 1 + \frac{(1 - e^{-G(f)}) |H(f)|^2 P(f)}{N(f)} \right) df
\]
for a given \( H(f) \) and \( N(f) \), subject to the constraints represented by (4) and (15).

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B. Optimal Solution

Proceeding as in [9, Chap. 8], it can shown that the above problem can be broken into an equivalent discrete optimization problem when \( N(f) \) is represented on an orthonormal basis using Karhunen-Loeve expansion. Let \( X_i = X(f_i) \) represent the \( i \)-th discrete sample of a continuous waveform \( X(f) \), then for the continuous real-valued function \( C \), we make use of the following property of Riemann integrals to write

\[
C = \lim_{n \to \infty} \max_{P, G} C_s(P, G) \triangle f_n
\]

where, \( P = [P_1, \ldots, P_n]^T \), \( G = [G_1, \ldots, G_n]^T \),

\[
C_s(P, G) = \frac{1}{n} \sum_{i=1}^{n} \ln \left( 1 + \frac{(1 - e^{-G_i})|H_i|^2P_i}{N_i} \right) \tag{17}
\]

and \( \triangle f_n = W/n \). Here, \((.)^T\) represents the transpose of a vector. Hence, it is equivalent to optimize (17) and extend the solution to the continuous case.

Now, consider the problem of maximizing

\[
F(P, G) = \sum_{i=1}^{n} \ln \left( 1 + \frac{(1 - e^{-G_i})P_i}{Y_i} \right)
\]

subject to

\[
\sum_{i=1}^{n} G_i \leq G_0, \sum_{i=1}^{n} P_i \leq P_0
\]

where, \( Y_i = N_i/|H_i|^2 \).

Note that although \( F \) is increasing and concave in both \( P \) and \( G \), it is not jointly-concave in \((P, G)\). The Karush-Kuhn-Tucker (KKT) conditions say that the necessary conditions for \( x = [x_1, \ldots, x_n]^T \) to minimize \( f(x) \) subject to constraints \( c_j(x) \geq 0, j = 1, \ldots, k \) are that:

\[
\frac{\partial}{\partial x_i} \left( f(x) - \sum_{j} \lambda_j c_j(x) \right) = 0, \quad i = 1, \ldots, n
\]

\[
c_j(x) \geq 0, \quad \lambda_j \geq 0
\]

\[
\lambda_j c_j(x) = 0, \quad j = 1, \ldots, k
\]

For our problem, \( x = [P, G]^T \) and we want to minimize \(-F(P, G)\) subject to the constraints:

\[
c_1(x) : [u_n, 0_n]^T x \leq P_0,
\]

\[
c_2(x) : [0_n, u_n]^T x \leq G_0,
\]

where, \( u_n \) and \( 0_n \) represent \( n \)-length vectors comprising all ones and zeros, respectively.

To proceed, we form the Lagrangian

\[
L(P, G) = -F(P, G) + \lambda \sum_{j} P_j + \mu \sum_{j} G_j
\]

which leads to the necessary conditions

\[
\frac{\partial F}{\partial P_i} - \lambda = 0, \quad \frac{\partial F}{\partial G_i} - \mu = 0,
\]

\[
0 \leq P_i \leq P_0, \quad 0 \leq G_i \leq G_0,
\]

\[
\lambda \geq 0, \quad \mu \geq 0,
\]

\[
\lambda P_i = \mu G_i = 0.
\]

These KKT conditions are met if there exist Lagrange multipliers \( \lambda \) and \( \mu \) such that

\[
\frac{\partial F}{\partial P_i} = \frac{(1 - e^{-G_i})}{Y_i + (1 - e^{-G_i})P_i} \begin{cases} = \lambda, & P_i > 0 \\ \leq \lambda, & P_i = 0 \end{cases}
\]

\[
\frac{\partial F}{\partial G_i} = \frac{e^{-G_i}P_i}{Y_i + (1 - e^{-G_i})P_i} \begin{cases} = \mu, & G_i > 0 \\ \leq \mu, & G_i = 0 \end{cases}
\]

Solving these, we get

\[
P_i = \left[ \frac{1}{\lambda} - \frac{Y_i}{1 - e^{-G_i}} \right]^+, \tag{18}
\]

\[
G_i = \left[ \ln (1 + \mu^{-1}) - \ln \left( 1 + \frac{Y_i}{P_i} \right) \right]^+. \tag{19}
\]

where, \( \lambda \) and \( \mu \) are chosen to satisfy the power and broadband matching constraints. It can seen that the optimal \( P_i \) and \( G_i \) satisfy a water-pouring solution, the continuous-frequency solution to which is given by

\[
P_{opt}(f) = \left[ 1 - \frac{N(f)}{|H(f)|^2(1 - e^{-G_{opt}(f)})} \right]^+, \tag{20}
\]

\[
G_{opt}(f) = \left[ \ln (1 + \mu^{-1}) - \ln \left( 1 + \frac{N(f)}{|H(f)|^2P_{opt}(f)} \right) \right]^+. \tag{21}
\]
VI. ITERATIVE MUTUAL-WATER-POURING ALGORITHM

Observe that the optimal spectrum must satisfy a water-pouring solution of the form given in (18), (19). The optimal solution presented above can easily be obtained by an iterative algorithm presented below.

Algorithm Iterative Mutual-Water-Pouring

**Input:** $P_{0}, G_{0}, i = \{1, 2, \ldots, n\}$ sorted in ascending order

**Output:** $P, G, C_{\text{opt}}, \lambda, \mu$

1. initialize $m = n, C_{\text{opt}} = 0$
2. while $m \neq 0$
3. do
4. $P = [P_{1}, \ldots, P_{m}], P_{i} = P_{0}/m$
5. $G = [G_{1}, \ldots, G_{m}], G_{i} = G_{0}/m$
6. $S = [S_{1}, \ldots, S_{m}], S_{i} = Y_{i}(1 - e^{-G_{i}})^{-1}$
7. $R = [R_{1}, \ldots, R_{m}], R_{i} = \ln (1 + Y_{i}/P_{i})$
8. $(P, \lambda) = \text{waterpour}(P, S, P_{0})$
9. $(G, \mu) = \text{waterpour}(G, R, G_{0})$
10. repeat 8 & 9
11. until $(P, G)$ converges to $(P_{\text{opt}}, G_{\text{opt}})$
12. evaluate $C_{s}$
13. if $C_{s} > C_{\text{opt}}$
14. update $C_{\text{opt}} = C_{s}, P_{\text{opt}}, G_{\text{opt}}, \lambda$ and $\mu$
15. $m = m - 1$
16. return

The function $\text{waterpour}(P, S, P_{0})$ distributes the total power $P_{0}$ such that $P_{i} = [\lambda^{-1} - S_{i}]^{+}$. This algorithm has been found to quickly converge to the optimal solution, typically requiring less than 10 iterations for $n = 100$. 

VII. RESULTS

We assume a simple resonant model [15] for the transmit and receive antennas: a series $RLC$ circuit. The impedance offered by the transmit (and receive) antenna is given by

$$Z_{A}(s) = R + Ls + \frac{1}{Cs}$$

The resonant radian-frequency $\omega_{0} = 2\pi f_{0}$ of the circuit is defined such that $Z_{a}(\omega_{0})$ is purely resistive: $\omega_{0} = 1/\sqrt{LC}$.

A resonant antenna with its quality-factor $Q$ is well modeled by a series $RLC$ circuit with $R = 1$, $L = Q/\omega_{0}$ and $C = 1/Q\omega_{0}$ [15]. The corresponding transmission and reflection coefficient is given, respectively, by

$$T(s) = \left(1 + \frac{Q}{2} \left(\frac{\omega_{0}}{s} + \frac{s}{\omega_{0}}\right)\right)^{-1}$$

and

$$\Gamma(s) = \frac{1 + (s/\omega_{0})^{2}}{1 + (s/\omega_{0})^{2} + 2(s/Q\omega_{0})}.$$  

Thus, the transmit antenna frequency response is

$$|T_{t}(f)|^{2} = 1 - |\Gamma_{t}(f)|^{2} = \frac{4f^{2}}{4f^{2} + Q^{2}(f^{2} - f_{0}^{2})^{2}}$$

The resonant radian-frequency $\omega_{0}$ is typically requiring less than 10 iterations for $n = 100$. Analysis of convergence for larger $n$ is under study.

and the receiver noise PSD in (13) is

$$N(f) = N_{\text{sky}}(f)(1 - |\Gamma(f)|^{2}) + N_{\text{amp}}(f)$$

where, $N_{\text{sky}}(f)$ represents the sky-noise and $N_{\text{amp}}(f)$ amplifier-noise. We however focus on amplifier-noise-dominant scenarios [16] such that $N(f) = N_{\text{amp}}(f)$ and it is assumed to be additive white Gaussian with variance $N_{0}$. Without loss of generality, it is convenient to work with the normalized frequency $f_{n} = f/f_{0}$ such that

$$\int_{W_{n}} P(f_{n})df_{n} \leq P_{n} = \frac{P_{0}}{f_{0}},$$

$$\int_{W_{n}} G(f_{n})df_{n} \leq G_{n} = \frac{G_{0}}{f_{0}}$$

with $W_{n}$ being the normalized bandwidth and $C_{n} = C/f_{0}$.

The normalized capacity, where

$$|H(f_{n})|^{2} = \alpha f_{n}^{-2}(1 - |\Gamma_{t}(f_{n})|^{2}), \quad \alpha = \left(\frac{C}{4\pi f_{0}^{2}f_{n}^{2}}\right)^{2}$$

and we set $\alpha = 1$.

The transmission coefficient (22) has a single zero at origin and a single zero at infinity. The corresponding reflection coefficient (23) has zeros $\lambda_{n} = \pm i\omega_{0}$ and poles

$$\lambda_{p} = \frac{\omega_{0}}{Q} \left(-1 \pm i\sqrt{Q^{2} - 1}\right).$$

Therefore, the set of matching constraints are given by

$$\int_{W_{n}} G(f_{n})df_{n} \leq \frac{2\pi}{Q} = G_{n}$$

and

$$\int_{W_{n}} f_{n}^{2} G(f_{n})df_{n} \leq \frac{4\pi^{2}}{Q} = 2\pi G_{n}.$$
A rather interesting result of our work is based on the fact that the capacities of optimal and sub-optimal water-pouring distribution are very close to each other when the bandwidth employed is optimum. Although, generalization to the MIMO case may not be trivial, we conjecture that the optimal matching-network characteristic would be a variant of the single antenna water-pouring solution in a multi-dimensional space.

VIII. CONCLUSION

We presented an information theoretic approach to characterize the optimal matching of a broadband communication system. We proposed an iterative algorithm for the optimal solution and demonstrated that a frequency-flat reflection coefficient over an arbitrary bandwidth is not optimal, although, such an approach with optimal bandwidth gives near-optimal results.

REFERENCES


REFERENCES

A. Simulation

Fig. 5 shows the result for an antenna with $Q = 25$. The other parameters used for the simulation are $P_n = 1$ and $N_0 = 13$ dB resulting in a peak SNR of 0 dB. The $P(f)$ shown in the figure is normalized such that $\max[P(f)] = 1$. The optimal bandwidth $W_o$ in this case is found to be 6.02%. A typical value for the bandwidth of an antenna is characterized by its 3 dB double-sided relative bandwidth $[17]$, which in this case amounts to $W_{\text{dSB}} = Q^{-1} = 4\%$. However, it is important to point out that since this is a joint optimization, $W_o$ is a function of SNR.

Fig. 5 also shows the performance for a frequency-flat reflection coefficient $\Gamma_{\text{uni}}(f) = e^{-G_0/2W_n}$ and the corresponding white $P_{\text{uni}}(f) = P_0/W_n$, when optimal bandwidth ($W_o = 6.02\%$) is employed. A nominal penalty of 0.70% in normalized capacity is observed compared to the optimal solution. One of the reason for this lies in the term $1 - e^{-G(f)}$ which for $G_0 \gg 1$ has a negligible impact on the capacity compared to a box-car $G(f) = G_0/W_o$.

To further quantify the sub-optimality of bandwidth for $(P_{\text{uni}}, \Gamma_{\text{uni}})$, see Fig. 6 which shows the loss in capacity relative to the optimal capacity for different values of $W_n$.

B. Discussion

For this example, the optimal $G(f)$ turns out to be such, that the second inequality constraint is inactive and hence, the Lagrangian used to find the optimal solution is correct. It can be shown that for a $G(f)$ symmetric about $f_0$, the second constraint is always inactive.

In real-world problems, the source or the load impedance may not always be modeled using a finite lumped element model. To circumvent this problem, there exist numerical methods such as the real-frequency technique [18]. It is important to point out that there may exist impedances which do not have a constraint like (15) to start with, however, a similar approach can be employed.

An overview of typical $Q$ values can be found in [17, Sec. 11.5] and [11, Sec. 8.2].