Universal Fuzzy Models and Universal Fuzzy Controllers Based on Generalized T-S Fuzzy Models

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Abstract—This paper investigates the universal fuzzy models problem and universal fuzzy controllers problem for discrete-time general nonlinear systems based on a class of generalized T-S fuzzy models. The generalized T-S fuzzy models, which are shown to be universal function approximators, are also proved to be universal fuzzy models for non-affine nonlinear systems under some sufficient conditions. The results of static and dynamic universal fuzzy controllers for two classes of nonlinear systems are then given, respectively, and constructive procedures to obtain the universal fuzzy controllers are also provided.

Index Terms—T-S fuzzy models; General nonlinear systems; Universal fuzzy models; Universal fuzzy controllers;

I. INTRODUCTION

Takagi and Sugeno (T-S) type fuzzy models [1] have been widely studied in recent decades and proved to be one of the most successful approaches to control design of complex nonlinear systems [2]-[8]. A T-S fuzzy model describes a nonlinear system by a set of fuzzy IF-THEN rules in the form of local linear or affine models which are smoothly connected by fuzzy membership functions. And this relatively simple and uniform structure demonstrates great advantages in stability analysis and controller design of fuzzy control systems in view of powerful conventional control theory. Due to the great efforts of researchers, a large volume of works on theoretical results of stability analysis and controller synthesis of T-S fuzzy systems have been reported in fuzzy control literature [1]-[8].

To control a nonlinear system, a typical way we followed can be summarized as two steps: first, for a given nonlinear system, construct its approximating T-S fuzzy model; then, design a controller for the obtained T-S fuzzy model. It is expected that the designed controller is able to stabilize the original nonlinear system if the approximation accuracy is good enough. However, as it is argued in [17], the commonly used T-S type fuzzy models, where the control variables are not included in the premise variables, are only able to approximate affine nonlinear systems to any degree of accuracy on any compact set. This implies that only the stabilization problem of affine nonlinear systems can be solved based on the commonly used T-S fuzzy models. To deal with control design problem of more general nonlinear system, we proposed a class of generalized T-S fuzzy models which were proved to be universal function approximators in [19]. However, an important question still needs to be investigated further. That is, whether the generalized T-S fuzzy models provide to be universal fuzzy models in the sense that the approximation error between states of the original nonlinear system and its approximating T-S fuzzy system can be made arbitrary small.

In [19], we have obtained some preliminary results on control design of general nonlinear systems based on generalized T-S fuzzy models. However, a critical question arises naturally. That is, given a complex nonlinear system which can be stabilized by an appropriately defined controller, does there exist a fuzzy controller to stabilize it? This is the so-called universal fuzzy controllers problem [9]-[14]. Furthermore, how can one design the fuzzy controller if it exists? In [13], the authors have analyzed the universal fuzzy control problem for a class of nonlinear systems, and a constructive procedure is provided to obtain the universal fuzzy controller by solving a set of nonlinear equations. In [14], it is shown that Mamdani-type fuzzy controllers are universal fuzzy controllers for nonlinear systems which are globally asymptotically stabilizable. A constructive method to obtain the universal fuzzy controller is also given. However, due to the approximation capability of commonly used T-S fuzzy models, when dealing with more
general nonlinear systems (non-affine nonlinear systems)[13]-
[14], fuzzy models in which the local models are nonlinear
functions of control variables are employed instead, and in
this case the universal fuzzy controller is obtained by solving
a set of nonlinear equations which might be a very difficult
task in general [13]-[14].

In this paper, we investigate the universal fuzzy models
problem and universal fuzzy controllers problem for discrete-
time general nonlinear systems based on the generalized T-
S fuzzy models [19]. The rest of this paper is organized as
follows. In section II, the generalized T-S fuzzy models
are introduced and proved to be universal fuzzy models for
general nonlinear systems under some sufficient conditions.
Then by using a class of dynamic fuzzy controllers, the results
of static and dynamic universal fuzzy controllers for a class
of exponentially stabilizable nonlinear systems and a class
of asymptotically stabilizable nonlinear systems are given in
section III and IV, respectively. Constructive procedures to
obtain the universal fuzzy controllers are also given in both
sections. Conclusions are given in section V.

II. DYNAMIC FUZZY MODELS AS UNIVERSAL FUZZY MODELS

Let us consider a general nonlinear system described by the
following equation,

\[ x(t + 1) = f(x(t), u(t)), \]

where \( x(t) = [x_1(t), ..., x_n(t)]^T \in X = \prod_{i=1}^n X_i \subset \mathbb{R}^n \) with
\( x_0 = [0, ..., 0]^T \in X \) and \( u(t) = [u_1(t), ..., u_m(t)]^T \in U = \prod_{j=1}^m U_j \subset \mathbb{R}^m \) with
\( u_0 = [0, ..., 0]^T \in U \). Throughout the paper, it is always
assumed that the origin is the equilibrium of the system, that is,
\( f(0, 0) = 0 \), and \( f(x, u) \) is a continuously differentiable function on
\( X \times U \). Further it is assumed that \( X \times U \) is a compact set on
\( \mathbb{R}^n \times \mathbb{R}^m \).

We consider to approximate the nonlinear system in (1) by
the following kind of generalized T-S fuzzy models which was
proposed in our previous work [19].

\[ \text{Plant rule } R^i : \text{IF } \begin{array}{l}
\text{IF } x_1(t) \text{ is } \mu^i_1 \\
\text{AND ... AND } x_n(t) \text{ is } \mu^i_n \\
\text{AND } u_1(t) \text{ is } \nu^i_1 \\
\text{AND ... AND } u_m(t) \text{ is } \nu^i_m,
\end{array} \text{ THEN } \\
x(t + 1) = A_i x(t) + B_i u(t), i \in \mathcal{L} := \{1, 2, ..., r\}, \]

where \( \mathcal{L} \) denotes the \( i \)th rule, \( r \) the total number of rules,
\( \mu^i_r \) and \( \nu^i_r \) the fuzzy sets, \( x(t) \in \mathbb{R}^n \) the state vector, \( u(t) \in \mathbb{R}^m \)
the input vector, and \( [A_i, B_i] \) the matrices of the \( i \)th local
model.

Via the commonly used fuzzy inference method, that is,
the center-average defuzzifier, product inference and singleton
fuzzyfier, the T-S fuzzy system in (2) can be expressed as
follows,

\[ x(t + 1) = \hat{f}(x(t), u(t)) \]

with

\[
\begin{align*}
\hat{f}(x(t), u(t)) &= \sum_{i=1}^r \mu_i(x, u)[A_i x(t) + B_i u(t)], \\
\mu_i(x, u) &= \frac{\prod_{j=1}^n \mu^i_j(x_j) \prod_{j=1}^m \nu^i_j(u_j)}{\sum_{i=1}^r \prod_{j=1}^n \mu^i_j(x_j) \prod_{j=1}^m \nu^i_j(u_j)},
\end{align*}
\]

where \( \mu_i(x, u) \) are the so-called normalized fuzzy membership
functions satisfying \( \mu_i(x, u) \geq 0 \) and \( \sum_{i=1}^r \mu_i(x, u) = 1 \).

It has been proved in [19] that the generalized T-S fuzzy
models in (2) are universal function approximators, which is
shown in the following lemma.

**Lemma 2.1.** [19] For any given function \( f(x, u) \in C^1 \)
on the compact set \( X \times U \) with \( f(0, 0) = 0 \) and any
positive constant \( \epsilon_f \), there exists a T-S fuzzy model \( \hat{f}(x, u) = \sum_{i=1}^r \mu_i(x, u)[A_i x(t) + B_i u(t)] \) given in (3) such that, for any
\( (x, u) \in X \times U \),

\[ \hat{f}(x, u) = f(x, u) + \epsilon(x, u), \]

and

\[ \|\epsilon(x, u)\| \leq \epsilon_f \|\tilde{x}\|, \]

where \( \tilde{x} = [x_1, ..., x_n, u_1, ..., u_m]^T \in \mathbb{R}^{n+m} \).

**Remark 2.1.** It has been argued in [19] that the norm bound of the approximation error \( \|\epsilon(x, u)\| \) can be arbitrarily small by choosing \( \epsilon_f > 0 \) accordingly. In other words, the fuzzy dynamic model given in (2) can approximate the general nonlinear system in (1) to any degree of accuracy on a compact set.

In fact, Lemma 2.1 provides a constructive procedure to
obtain the generalized T-S model with the aid of the following
fact.

**Fact 2.1.** The matrix function \( Q(x) = [Q^i_0(x)] \in \mathbb{R}^{n \times n} \) where \( Q^i_0(x) = \sum_{j=1}^{r_0} \mu^i_0(x) q^i_j, \mu^i_0(x) \geq 0 \) and
\( \sum_{j=1}^{r_0} \mu^i_0(x) = 1 \), can be rewritten as \( Q(x) = \sum_{i=1}^{r_0} \nu_i(x) Q_i \)
where \( Q_i \in \mathbb{R}^{n \times n}, \nu_i(x) \geq 0 \) and \( \sum_{i=1}^{r_0} \nu_i(x) = 1 \).

**Proof.** The proof is omitted here.

One has the following constructive algorithm.

**Algorithm 2.1.** Given a nonlinear system in the form
(1) which is denoted by \( x(t + 1) = F(x(t), u(t)) \), one can construct a generalized T-S fuzzy system \( x(t + 1) = F^i(x(t), u(t)) = \sum_{i=1}^{r_0} \mu_i(x, u)[C_i x(t) + D_i u(t)] \) to approximate \( F(x(t), u(t)) \) by the following steps.

**Step 1.** By using Lemma A1 in the appendix, transform \( F(x(t), u(t)) \) into the product of \( F^i(\tilde{x}(t)) \) and \( \tilde{x}(t) \), where
\( F^i(\tilde{x}(t)) = [F^i_0(\tilde{x}(t))] \in \mathbb{R}^{n \times (m+n)}, i \in \mathcal{J} = \{1, ..., n\}, j \in \mathcal{J} \) \( \cup \{1, ..., m+n\} \) is a continuous function and \( \tilde{x} = [x_1, ..., x_n, u_1, ..., u_m]^T \in \mathbb{R}^{n+m} \).

**Step 2.** By using the approximation schemes given in
[15] or [16], construct the Type II fuzzy model [7], that is,
\( F^i_0(\tilde{x}(t)) \) of each element \( F^i_0(\tilde{x}(t)) \). Then the corresponding fuzzy model for \( F^i(\tilde{x}(t)) \) is denoted by \( \tilde{F}^i(\tilde{x}(t)) = [\tilde{F}^i_0(\tilde{x}(t))] \). Without loss of generality, it is assumed that each fuzzy model \( \tilde{F}^i_0(\tilde{x}(t)) \) has \( r_0 \) rules. Suppose \( \tilde{F}^i_0(\tilde{x}(t)) = \sum_{i=1}^{r_0} \tilde{\mu}_i(\tilde{x}(t)) q^i_j, \tilde{\mu}_i(\tilde{x}(t)) \) are normalized membership functions and \( q^i_j \) are positive constants.

**Step 3.** By using Fact 2.1, rewrite \( \tilde{F}^i(\tilde{x}(t)) \) as \( \tilde{F}^i(\tilde{x}(t)) = \sum_{i=1}^{r_0} \nu_i(x, u) Q_i \).

In this way, one can construct the generalized T-S fuzzy
model for \( x(t + 1) = F(x(t), u(t)) \), that is, \( x(t + 1) = \tilde{F}(x(t), u(t)) = \tilde{F}(\tilde{x}(t)) = \sum_{i=1}^{r_0} \mu_i(x, u)[C_i x(t) + D_i u(t)] \)
\( \dot{D}_t u(t) \), where \( r_1 = r_0^{(m+n) \times n} \), \( \mu_t(x, u) = \nu_t(\bar{x}) \) and \( Q_t = [C_t, D_t] \).

**Remark 2.2.** It should be noted that the result in Lemma 2.1 only answers the approximation problems between two static nonlinear functions, that is, \( f(x, u) \) and \( \hat{f}(x, u) \). However, the approximation errors between the states of two dynamic systems, that is, systems described in (1) and (3), might grow as time goes. Much care should be taken in dealing with the systems, that is, systems described in (1) and (3), might grow.

Theorem 2.1. \( \tilde{f}(x, t) - x(t) \) for all \( \varepsilon > 0 \) and any \( f \in SS \), there exists a TS fuzzy model \( \hat{f} \in SS \) such that for the two dynamic systems (1) and (7) under the same initial condition,

\[
\sup_{t \geq 0} \| \tilde{f}(t) - x(t) \| < \varepsilon.
\]

Here we first introduce a lemma.

**Lemma 2.2. (Discrete Gronwall Inequality)**[20] Assume \( \langle x_n \rangle, \langle p_n \rangle, \) and \( \langle q_n \rangle \) are nonnegative sequences and

\[
x_n \leq p_n + \sum_{k=0}^{n-1} q_k x_k, \text{ for } n \geq 0.
\]

Then

\[
x_n \leq p_n + \sum_{k=0}^{n-1} p_k q_k \prod_{j=k+1}^{n-1} (1 + q_j), \text{ for } n \geq 0.
\]

For the nonlinear systems in (1), we denote

\[
\frac{\partial f(x, u)}{\partial \langle x^T, u^T \rangle} \bigg|_{\langle x, u \rangle = 0} = [A, B],
\]

where \( A \in \mathbb{R}^{n \times n} \) and \( B \in \mathbb{R}^{n \times m} \).

Furthermore, we denote \( \hat{b}(x, u) = f(x, u) - Ax \) and \( \hat{b}(x, u) = f(x, u) - Ax \). Since \( \hat{b}(x, u) \) and \( \hat{b}(x, u) \) are both continuously differentiable, \( \hat{b}(\cdot) \) satisfies the Lipschitz condition, i.e., there exists a positive constant \( \beta_b \) such that

\[
\| \hat{b}(x_1, u) - \hat{b}(x_2, u) \| \leq \beta_b \| x_1 - x_2 \|.
\]

Then we have the following result.

**Theorem 2.1.** \( \hat{f}(x, t) - x(t) \) for all \( \varepsilon > 0 \) and any \( f \in SS \), there exists a TS fuzzy model \( \hat{f} \in SS \) such that for the two dynamic systems (1) and (7) under the same initial condition,

\[
\sup_{t \geq 0} \| \tilde{f}(t) - x(t) \| < \varepsilon.
\]

By using Lemma 2.1, given any constant \( \varepsilon_b > 0 \), one can obtain the following generalized TS fuzzy model,

\[
\hat{x}(t + 1) = \hat{f}(\hat{x}(t), u(t)) = A\hat{x}(t) + \hat{b}(\hat{x}(t), u(t)),
\]

such that

\[
\| \hat{b}(x(t), u(t)) - b(x(t), u(t)) \| < \varepsilon_b.
\]

Denote \( e(t) = \hat{x}(t) - x(t) \), then

\[
e(t + 1) = A\hat{e}(t) + E(t),
\]

where

\[
E(t) = \hat{b}(x(t), u(t)) - b(x(t), u(t)).
\]

And one has that,

\[
\| E(t) \| \leq \| \hat{b}(x(t), u(t)) - \hat{b}(x(t), u(t)) \| + \| \hat{b}(x(t), u(t)) - b(x(t), u(t)) \| \\
\leq \beta_b \| e(t) \| + \varepsilon_b.
\]

Then,

\[
\| e(t) \| \leq \frac{\beta_b}{\beta_b - \| A \|} \| e(t-1) \| + \frac{\varepsilon_b}{\beta_b - \| A \|}.
\]

For any \( \varepsilon > 0 \), by choosing \( \varepsilon_b = \varepsilon \), we can conclude that

\[
\| e(t) \| \leq \varepsilon, \forall t \geq 0,
\]

which completes the proof.

Based on Theorem 2.1, we also have the following corollaries.

**Corollary 2.1.** \( \hat{f} \) are universal fuzzy models for exponentially stable nonlinear systems in form of (1) on a sufficient small compact set around the origin.

**Corollary 2.2.** \( \hat{f} \) are universal fuzzy models for asymptotically stable nonlinear systems in form of (1) on a sufficient small compact set around the origin, if the matrix \( A \), which is defined in (11), has no eigenvalues equal to 1.

III. UNIVERSAL FUZZY CONTROLLERS FOR A CLASS OF NONLINEAR SYSTEMS

In [19], we have proposed an approach to stabilization of the general nonlinear systems in (1) via its generalized TS fuzzy models, by using a class of dynamic feedback controllers. In this section, we will study the so-called universal fuzzy controller problem [9], that is, if a nonlinear system in (1) can be stabilized by a dynamic feedback controller described by
\[ u(t+1) = g(x(t), u(t)), \] does there exist a generalized T-S type fuzzy controller to stabilize the nonlinear system?

Let \( FC \) be the set of all generalized T-S type fuzzy controllers of the following form.

**Plant rule \( \mathcal{R}_i \):** IF \( x_1(t) \) is \( \mu_{i1}^1 \) AND ... AND \( x_n(t) \) is \( \mu_{in}^l \) ; \( u_1(t) \) is \( \nu_{11}^1 \) AND ... AND \( u_m(t) \) is \( \nu_{m1}^l \); THEN

\[ u(t+1) = F_l x(t) + G_l u(t); l \in \mathcal{L} := \{1, 2, \ldots, r\} \quad (23) \]

which can be rewritten via the standard fuzzy blending as

\[ u(t+1) = \sum_{l=1}^{r} \mu_l(x, u)[F_l x(t) + G_l u(t)]. \quad (24) \]

First we introduce the following definitions.

**Definition 3.1.** Any \( f \in \mathcal{SS} \) is said to be semi-globally uniformly exponentially stabilizable on a compact set \( X \times U \subset \mathbb{R}^n \times \mathbb{R}^m \) which contains the equilibrium, if there exists a dynamic feedback control law \( u(t+1) = g(x(t), u(t)) \) such that the closed-loop system

\[
\begin{align*}
&x(t+1) = f(x(t), u(t)) \\
&u(t+1) = g(x(t), u(t))
\end{align*}
\]

(25)

is semi-globally uniformly exponentially stable on a compact set \( X \times U \subset \mathbb{R}^n \times \mathbb{R}^m \), that is, there exist positive constants \( C, 0 < \sigma < 1 \) and \( \lambda > 0 \) and a region \( X_0 \times U_0 \subset X \times U \), such that given any initial states \( (x(0), u(0)) \in X_0 \times U_0 \), the solution \((x(t), u(t)) \) of (25) exists for all \( t \geq 0 \) and satisfies

\[
\|x(t)^T, u(t)^T\|^T \leq C\|x(0)^T, u(0)^T\|^T \|a^T. \quad (26)
\]

**Definition 3.2.** Any \( f \in \mathcal{SS} \) is said to be globally uniformly exponentially stabilizable if all the conditions in Definition 3.1 hold globally.

**Definition 3.3.** \( FC \) are said to be static universal fuzzy controllers, if for any \( f \in \mathcal{SS} \) which is globally uniformly exponentially stabilizable there exists a dynamic feedback fuzzy control law \( \hat{g}(x, u) \in \mathcal{FC} \) such that the closed-loop control system

\[
\begin{align*}
&x(t+1) = f(x(t), u(t)) \\
&u(t+1) = \hat{g}(x(t), u(t))
\end{align*}
\]

(27)

is semi-globally uniformly exponentially stable on a compact set \( X \times U \subset \mathbb{R}^n \times \mathbb{R}^m \).

**Definition 3.4.** [13] \( FC \) are said to be dynamic universal fuzzy controllers, if they are static universal fuzzy controllers and the approximation error between the states of two dynamic systems (25) and (27) can be made arbitrary small.

Then we are ready to present the main results of this section.

**Theorem 3.1.** \( FC \) are static universal fuzzy controllers for a class of nonlinear systems which belong to \( \mathcal{SS} \) and are globally uniformly exponentially stabilizable.

**Proof.** Since \( f \in \mathcal{SS} \) is globally uniformly exponentially stabilizable, there exists a control law \( u(t+1) = g(x(t), u(t)) \) in \( C^1 \) such that the closed-loop system given as in (25) is globally uniformly exponentially stable.

Denote \( \bar{x} = [x_1, ..., x_n, u_1, ..., u_m]^T \in \bar{X} \subset \mathbb{R}^{m+n} \).

According to Lemma 2.1, for a given small enough constant \( \epsilon_g > 0 \), one can find a fuzzy control law \( u(t+1) = \hat{g}(x(t), u(t)) \in \mathcal{FC} \) such that

\[
\hat{g}(x(t), u(t)) = g(x(t), u(t)) + \epsilon(x(t), u(t)), \quad (28)
\]

where

\[
\|\epsilon(x, u)\| \leq \epsilon_g \|\bar{x}\|. \quad (29)
\]

Thus the closed-loop control system (27) can be rewritten as

\[
\begin{align*}
&\bar{x}(t+1) = f(x(t), u(t)), \\
&u(t+1) = \hat{g}(x(t), u(t)) + \epsilon(x, u).
\end{align*}
\]

(30)

For the sake of simplicity, rewrite (25) as

\[
\bar{x}(t+1) = \hat{G}(\bar{x}(t)) = G(\bar{x}(t)) + \bar{\epsilon}(\bar{x}), \quad (31)
\]

and (30) as

\[
\left\{ \begin{array}{l}
\bar{x}(t+1) = \hat{G}(\bar{x}(t)) = G(\bar{x}(t)) + \bar{\epsilon}(\bar{x}), \\
\end{array} \right.
\]

(32)

By the Lyapunov converse theorem proposed in [22], if the system in (31) is globally uniformly exponentially stable then there exist a smooth Lyapunov function \( V(\bar{x}(t)) \), and some positive constants \( c_1 \) and \( c_2 \) such that

\[
\|\bar{x}\| \leq V(\bar{x}(t)) \leq c_1 \|\bar{x}(t)\|, \quad (34)
\]

\[
V(G(\bar{x}(t))) \leq V(\bar{x}(t)) \leq c_2 \|\bar{x}\|. \quad (35)
\]

In addition, it can be also concluded from [22] that, for any function \( d(t) \in \Omega \), where \( \Omega \) is a compact set, one has that

\[
\|V(\bar{x}(t) + d) - V(\bar{x}(t))\| \leq c_3 \|d\|. \quad (36)
\]

The difference of this Lyapunov function along the trajectories of the system (32) satisfies

\[
\begin{align*}
&V(G(\bar{x}(t)) + \bar{\epsilon}(\bar{x}(t))) - V(\bar{x}(t)) \\
&= V(G(\bar{x}(t))) + \bar{\epsilon}(\bar{x}(t))) - V(G(\bar{x}(t))) \\
&= V(G(\bar{x}(t))) - V(\bar{x}(t)) \\
&\leq c_3 \|\bar{\epsilon}(\bar{x}(t))\| - c_2 \|\bar{x}(t)\| \\
&\leq (c_3 \epsilon_g - c_2) \|\bar{x}(t)\|.
\end{align*}
\]

(37)

Thus if one chooses a fuzzy control law such that \( \epsilon_g < c_2/c_3 \), it follows from (37) that

\[
\dot{V}(\bar{x}(t)) < -\bar{c} \|\bar{x}(t)\|, \quad (38)
\]

\[
\bar{c} = c_2 - c_3 \epsilon_g.
\]

Then one can conclude that (32) is semi-globally uniformly exponentially stable on the compact set \( \bar{X} \), or equivalently, the closed-loop control system (27) is semi-globally uniformly exponentially stable on the compact set \( X \times U \). Thus \( FC \) are static universal fuzzy controllers.

**Remark 3.1.** For details of the obtained Lyapunov function which satisfies (34)-(36), one can refer to [22].
Theorem 3.2. FC are dynamic universal fuzzy controllers for a class of nonlinear systems which belong to $SS$ and are globally uniformly exponentially stabilizable.

Proof. Theorem 3.2 is a direct result from Corollary 2.1, and the proof is omitted here.

If a reference model $\tilde{x}(t+1) = G_m(\tilde{x}(t)) = [f(x(t), u(t))^T, g_m(x(t), u(t))^T]^T$ is given, one can apply Algorithm 2.1 to obtain the model reference fuzzy controller. That is, one can construct a fuzzy control law $\hat{g}(x, u) \in FC$ such that for any given $\epsilon_m > 0$,

$$\hat{g}(x, u) = g_m(x, u) + \epsilon(x, u),$$ (39)

where

$$\|\epsilon(x, u)\| \leq \epsilon_m\|\tilde{x}\|,$$ (40)

and the closed-loop control system

$$\tilde{x}(t+1) = \hat{G}(\tilde{x}(t)) = G_m(\tilde{x}(t)) + \hat{\epsilon}(\tilde{x}),$$ (41)

where $\hat{G}(\tilde{x}(t)) = [f(x(t), u(t))^T, \hat{g}(x(t), u(t))^T]^T$ and $\hat{\epsilon}(x) = [0, \epsilon(x, u)^T]^T$, is semi-globally uniformly exponentially stable on the compact set $\tilde{X}$.

IV. Universal Fuzzy Controllers for More General Nonlinear Systems

In section III, we have shown that the fuzzy controllers defined in (24) are both static and dynamic universal fuzzy controllers for nonlinear systems which are uniformly exponentially stabilizable. In this section, we will consider more general nonlinear systems which are only globally asymptotically stabilizable.

For the sake of simplicity, we use the simplified forms as in (31) and (32) to represent the closed-loop control systems given as in (25) and (27), respectively.

We first introduce the following definitions.

Definition 4.1. Any $f \in SS$ is said to be semi-globally asymptotically stabilizable on a compact set $X \times U \subset \mathbb{R}^n \times \mathbb{R}^m$ which contains the equilibrium, if there exists a control law $u(t+1) = g(x(t), u(t))$ such that the closed-loop system given by (31) is semi-globally asymptotically stable on a compact set $\tilde{X} \subset \mathbb{R}^{m+n}$, that is, there exist a $KL$ function $\beta$ and a region $\tilde{X}_0 \subset \tilde{X}$, such that given any initial states $\tilde{x}(0) \in \tilde{X}_0$ the solution $\tilde{x}(t)$ of (31) exists for all $t \geq 0$ and satisfies

$$\|\tilde{x}(t)\| \leq \beta(||\tilde{x}(0)||, t).$$ (42)

Definition 4.2. Any $f \in SS$ is said to be globally asymptotically stabilizable if all the conditions in Definition 4.1 hold globally.

Definition 4.3. FC are said to be static universal asymptotic fuzzy controllers, if for any $f \in SS$ which is globally asymptotically stabilizable there exists a dynamic feedback fuzzy control law $\hat{g}(x, u) \in FC$ such that the closed-loop system (32) is semi-globally asymptotically stable on a compact set $\tilde{X} \subset \mathbb{R}^{m+n}$.

Definition 4.4. FC are said to be dynamic universal asymptotic fuzzy controllers, if they are static universal fuzzy controllers and the approximation error between the states of two dynamic systems (31) and (32) can be made arbitrary small.

Before presenting the other main result of this section, we first introduce a lemma.

Lemma 4.1 (Discrete-time Converse Lyapunov Theorem) [21] If $G(\tilde{x}(t))$ is globally asymptotically stable then there exist a Lyapunov function $V(\tilde{x}(t))$, a $K_\infty$ function $\alpha_1(\cdot)$, a $K_\infty$ function $\alpha_2(\cdot)$ and $K$ function $\alpha_3(\cdot)$ such that

$$\alpha_1(||\tilde{x}(t)||) \leq V(\tilde{x}(t)) \leq \alpha_2(||\tilde{x}(t)||),$$ (43)

$$V(G(\tilde{x}(t))) - V(\tilde{x}(t)) \leq -\alpha_3(||\tilde{x}(t)||).$$ (44)

In addition, for any function $d(t) \in \Omega$, where $\Omega$ is a compact set, one has that

$$||V(\tilde{x}(t) + d) - V(\tilde{x}(t))|| \leq c||d||,$$ (45)

where $c$ is a positive constant.

Theorem 4.1. FC are static universal fuzzy asymptotic controllers for the plants described in (1) which are globally asymptotically stabilizable, if for the $K$ function $\alpha_3(\cdot)$ given in (44) there exist a $K$ function $\alpha_4(\cdot)$ and a positive constant $\gamma$ such that

$$\inf_{||\tilde{x}(t)|| > 0, \tilde{x}(t) \in \tilde{X}} \frac{\alpha_3(||\tilde{x}(t)||) - \alpha_4(||\tilde{x}(t)||)}{||\tilde{x}(t)||} \geq \gamma.$$ (46)

Proof. Similar to the proof procedure of Theorem 4.1, for the Lyapunov function satisfying (43)-(45), if we choose a fuzzy control law $u(t+1) = \hat{g}(x(t), u(t)) \in FC$ such that

$$\hat{g}(x, u) = g(x, u) + \epsilon(x, u),$$ (47)

where

$$||\epsilon(x, u)|| \leq \epsilon_g ||\tilde{x}||,$$ (48)

and $\epsilon_g \leq \frac{\gamma}{\alpha_3}$.

The difference of this Lyapunov function along the trajectories of the system (32) satisfies

$$V(G(\tilde{x}(t)) + \epsilon(\tilde{x}(t))) - V(\tilde{x}(t))$$

$$= V(G(\tilde{x}(t))) + \epsilon(\tilde{x}(t))) - V(G(\tilde{x}(t)))$$

$$+ V(G(\tilde{x}(t))) - V(\tilde{x}(t))$$

$$\leq c||\tilde{x}(t)|| - \alpha_3(||\tilde{x}(t)||)$$

$$\leq c\epsilon_g ||\tilde{x}(t)|| - \alpha_3(||\tilde{x}(t)||)$$

$$\leq -\alpha_4(||\tilde{x}(t)||).$$ (49)

From the Lyapunov theorem in [21], one can conclude that the closed-loop system (47) is semi-globally asymptotically stable on the compact set $\tilde{X}$. Thus FC are universal asymptotic fuzzy controllers.

Remark 4.1. It is easily observed that, for the class of nonlinear systems discussed in section III, the condition (46) always holds.

Theorem 4.2. FC are dynamic universal asymptotic fuzzy controllers for the plants described in (1) which are globally asymptotically stabilizable, if the conditions in Theorem 4.1 are satisfied and the matrix $A$, which is defined in (11), has no eigenvalues equal to 1.
Proof. Theorem 4.2 is a direct result from Corollary 2.2 and the proof is omitted here.

If a reference model \( \ddot{x}(t + 1) = G_m(\ddot{x}(t)) = [f(x(t), u(t))]^T, g_m(x(t), u(t))^T]^T \) is given, one can apply Algorithm 2.1 to obtain the model reference fuzzy controller. That is, one can construct a fuzzy control law \( \hat{g}(x, u) \in FC \) such that for any given \( \epsilon_m > 0 \),

\[
\hat{g}(x, u) = g_m(x(u) + \epsilon(x, u), (50)
\]

and the closed-loop control system

\[
\dot{x}(t + 1) = \dot{G}(\dot{x}(t)) = G_m(\dot{x}(t)) + \epsilon(\ddot{x}),
\]

where

\[
\|\epsilon(x, u)\| \leq \epsilon_m \|\ddot{x}\|,
\]

and the generalized T-S type fuzzy controllers for general nonlinear systems under some sufficient conditions. It is also shown that the generalized T-S fuzzy models are universal fuzzy models for general nonlinear systems under some sufficient conditions. It is also shown that the generalized T-S type fuzzy controllers are both static and dynamic universal (or asymptotic) fuzzy controllers for nonlinear systems which are uniformly exponentially stabilizable or globally asymptotically stable on the compact set \( \bar{x} \in \mathbb{R}^{m+n} \).

V. CONCLUSIONS

In this paper, some new results on the universal fuzzy models and universal fuzzy controllers problem based a class of generalized T-S fuzzy models are presented. It is shown that the generalized T-S fuzzy models are universal fuzzy models for general nonlinear systems under some sufficient conditions. A class of nonlinear systems is uniformly exponentially stabilizable on the compact set \( \bar{x} \in \mathbb{R}^{m+n} \). Constructive procedures to obtain the universal fuzzy controllers are also provided.

APPENDIX

Lemma A1. [17] If a vector value function \( f(z) = [f_1(z_1, \ldots, z_N), \ldots, f_N(z_1, \ldots, z_N)]^T \) is continuously differentiable on \( Z = \prod_{i=1}^{N}[a_i, b_i] \) with \( 0 \in Z \) and \( f(0) = 0 \), then for \( i = 1, \ldots, N \), the vector value function

\[
G_i(z) = g_i(z_i, \ldots, z_N) = \begin{cases} f(0, \ldots, 0, z_i, \ldots, z_N) - f(0, \ldots, z_i, \ldots, z_N), & z_i \neq 0 \\ \frac{\partial f(0, \ldots, 0, z_i, \ldots, z_N)}{\partial z_i}, & z_i = 0 \end{cases}
\]

is continuous on \( Z \) and

\[
f(z) = \sum_{i=1}^{N} G_i(z_i) = \sum_{i=1}^{N} g_i(z_i, \ldots, z_N)z_i.
\]

REFERENCES


