A Monotonicity Index for the Monotone Fuzzy Modeling Problem

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Abstract—In this paper, the problem of maintaining the (global) monotonicity and local monotonicity properties between the input(s) and the output of an FIS model is addressed. This is known as the monotone fuzzy modeling problem. In our previous work, this problem has been tackled by developing some mathematical conditions for an FIS model to observe the monotonicity property. These mathematical conditions are used as a set of governing equations for undertaking FIS modeling problems, and have been extended to some advanced FIS modeling techniques. Here, we examine an alternative to the monotone fuzzy modeling problem by introducing a monotonicity index. The monotonicity index is employed as an approximate indicator to measure the fulfillment of an FIS model to the monotonicity property. It allows the FIS model to be constructed using an optimization method, or be tuned to achieve a better performance, without knowing the exact mathematical conditions of the FIS model to satisfy the monotonicity property. Besides, the monotonicity index can be extended to FIS modeling that involves the local monotonicity problem. We also analyze the relationship between the FIS model and its monotonicity property fulfillment, as well as derived mathematical conditions, using the Monte Carlo method.

Keywords—Fuzzy inference system; monotonicity property; monotonicity index; the sufficient conditions; monte carlo; evolutionary computation optimization; system identification

I. INTRODUCTION

Consider a fuzzy inference system (FIS) model, \( y = f(x; \theta) \), where \( (x = x_1, x_2, \ldots, x_n) \) is a vector and is simplified as \( x_k \), \( k = 1, 2, 3, \ldots, n \), and \( \theta \) is a parameter vector that describes the FIS model. \( \theta \) parameters the fuzzy membership functions (MFs), fuzzy rule base, consequents, and etc. On one hand, for an FIS model that fulfills the monotonicity property between its output, \( y \), with respect to its \( i \)-th input (where \( i \in \mathbb{N} \)), \( x_i \), within the universe of discourse, \( y \) monotonically increases or decreases as \( x_i \) increases, i.e., \( f(x_{kx}, x_i^{1}) \leq f(x_{kx}, x_i^{2}) \) or \( f(x_{kx}, x_i^{1}) \geq f(x_{kx}, x_i^{2}) \) respectively, for \( x_i^{1} < x_i^{2} \) within the universe of discourse. On the other hand, for an FIS model that fulfills the local monotonicity property between \( y \) with respect to its \( x_i \) within the upper and lower bounds of \( x_i \) and \( x_i^{\pm} \), respectively (where the limit between the upper and lower bound is a subset of the universe discourse), \( y \) monotonically increases or decreases as the \( i \)-th input, i.e., \( x_i \), increases, i.e., \( f(x_{kx}, x_i^{1}) \leq f(x_{kx}, x_i^{2}) \) or \( f(x_{kx}, x_i^{1}) \geq f(x_{kx}, x_i^{2}) \) respectively, for \( x_i^{1} < x_i^{2} \) within the limit.

Recently, the importance of the monotonicity property in fuzzy modeling has been highlighted in a number of publications [1-9]. The monotonicity property can be exploited as an additional qualitative information when the number of data samples is small or the fuzzy rule set is incomplete [8-9]. Besides, taking the additional qualitative information/knowledge of the system into consideration makes the model identification procedure less vulnerable to noise and inconsistencies in data samples, as well as mitigates the over-fitting phenomenon [8]. However, there are only a few articles that address the issues related to how to design a monotonicity-preserving FIS model [2]. As explained in [4], this has been viewed as a difficult problem in fuzzy modeling.

In this paper, the problem of maintaining the monotonicity and local monotonicity properties between the input(s) and the output of an FIS model is addressed as the monotone fuzzy modeling problem. It includes the construction of a type-1 or type-2 FIS model either manually or with machine learning techniques to generate/tune/optimize an FIS model [10] and to develop a new mathematical framework to facilitate/complement some FIS modeling procedures (e.g., similarity reasoning [11], re-labeling [12], operator and etc), by considering the monotonicity/local monotonicity relationship among the input(s) and the output of a system as additional qualitative information (or prior knowledge).

Recent investigations on the monotone fuzzy modeling problem aim to develop a set of mathematical conditions as the governing equations [1, 4, 8, 9], to apply the developed mathematical conditions to real-world problems [5-7], and to further extend or synthesize the mathematical conditions to/or with some advanced FIS modeling techniques [2, 10-11]. From the literatures, the mathematical conditions or guidelines for the Sugeno FIS model [1], Mamdani FIS model [8], and SIRM FIS model [4] have been developed. It is possible to use the mathematical conditions as a set of constraints to facilitate/govern a data-driven FIS modeling procedure, e.g., neuro-fuzzy modeling, or Evolutionary Computation (EC) fuzzy modeling. In our previous work [11], a monotonicity index has been proposed, and has been used as a method to approximately evaluate the monotonicity property of an FIS model before it is implemented in practice. The monotonicity index serves as a useful indicator to measure whether an FIS model fulfills the monotonicity property, or to what extent it fulfills the monotonicity property. Besides, to the best of our
knowledge, no work on the local monotonicity property for FIS modeling has been reported.

Another recent advancement is with regards to the evolutionary fuzzy system, i.e., how to use an EC technique, e.g. the Genetic algorithm (GA), for designing FIS models and for providing them with learning and adaptation capabilities [13]. In [13], a comprehensive review on the GA-based FIS model and the roles of the GA in FIS models have been provided. However, a search in the literatures reveals that exploitation of the monotonicity property and the local monotonicity property in EC-based FIS models is new. Besides, it is unsure how the monotonicity property can be exploited as an additional qualitative information.

In this paper, we demonstrate that the monotonicity property is a useful qualitative information for EC-based FIS models. We further propose an alternative for the monotone fuzzy modeling problem that is based on the monotonicity index. Instead of relying on the derived set of mathematical conditions, the monotonicity index measurement is used as a useful information/feedback signal to allow an FIS model to be constructed. This approach can be applied to various types of FIS modeling techniques. This line of research is important as it contributes towards three major outcomes. First, it allows the monotonicity property of an FIS model to be studied or confirmed based on simulations and/or experiments. It also enables the region beyond the monotonicity property to be studied, via the Monte Carlo simulation. Second, it provides a general method to construct a monotonicity-preserving FIS model, and allows the monotonicity property of an FIS model to be preserved, even if the mathematical conditions are not satisfied. The proposed approach allows the relationship between the developed mathematical derivations and the exact monotonicity conditions (which is unknown) to be studied, via the Monte Carlo simulation. Third, it offers a general method to construct a monotonicity-preserving FIS model, allowing the monotonicity property of an FIS model to be preserved, even if the mathematical conditions are not satisfied. The proposed approach allows the relationship between the developed mathematical derivations and the exact monotonicity conditions (which is unknown) to be studied, via the Monte Carlo simulation. Third, it offers a general method to construct a monotonicity-preserving FIS model, allowing the monotonicity property of an FIS model to be preserved, even if the mathematical conditions are not satisfied. The proposed approach allows the relationship between the developed mathematical derivations and the exact monotonicity conditions (which is unknown) to be studied, via the Monte Carlo simulation.

II. A REVIEW ON THE MONOTONICITY INDEX

In this paper, the fulfillment of the monotonicity property is measured or evaluated by comparing the output pairs of an FIS model [10]. The monotonicity index serves as an approximate indicator whether an FIS model observes the monotonicity property or not. From the literature review, the importance of the monotonicity test has been shown in [14]. It is required to distinguish monotone functions from functions that are far from monotone. The procedure is as follows.

Let \( y = f(x) \) be represented by an \( n \)-dimension matrix \( y_{x_k} \). \( y_{x_k} \) is the output of \( f(x_k) \), where the input(s) vector is \( x_k = [x_1, x_2, ..., x_n] \), \( k = 1, 2, ..., n, i < k \). The test to measure the fulfillment of the monotonicity property between \( y \) and \( x_i \) is as follows.

(A) Determine the upper and lower limits of the universe of discourse for the input(s), i.e., \( x_k \), and denote them as \( \bar{x}_k \) and \( x_k \), respectively.

(B) Divide \( x_k \) into \( n_k^l \) divisions. The grid size of \( x_k \), \( g_s = \frac{(\bar{x}_k - x_k)}{n_k^l} \). \( y_{x, x_2, ..., x_n} \) is denoted as an \( n_1^l \times n_2^l \times ... \times n_k^l \) matrix.

(C) Compare \( y_{x_k = 1, 2, 3, ..., n_k^l} \) and \( y_{x_k = 0, 1, 2, 3, ..., n_k^l} \), with a function which is denoted as monotone \( \text{monotone} (y_{x_k = 1, 2, 3, ..., n_k^l}) \).

Equation (1) or (2) is adopted for a monotonic increasing or decreasing relationship, respectively.

\[
\text{monotone} (y_{x_k = 1, 2, 3, ..., n_k^l}) = \begin{cases} 
1 & \text{if } y_{x_k = 1, 2, 3, ..., n_k^l} \leq y_{x_k = 0, 1, 2, 3, ..., n_k^l} \\
0 & \text{otherwise} 
\end{cases}
\]

Equation (3) is adopted for a monotonic increasing or decreasing relationship, respectively.

\[
\text{monotone} (y_{x_k = 1, 2, 3, ..., n_k^l}) = \begin{cases} 
1 & \text{if } y_{x_k = 1, 2, 3, ..., n_k^l} \geq y_{x_k = 0, 1, 2, 3, ..., n_k^l} \\
0 & \text{otherwise} 
\end{cases}
\]

(D) Obtain the Monotonicity Index between \( y \) and \( x_i \) using

\[
\text{Monotonicity Index} (y, x_i) = \sum_{x_k = 1}^{n_k^l} \left( \text{monotone} (y_{x_k = 1, 2, 3, ..., n_k^l}) \right) \frac{1}{n_k^l} 
\]

III. MONTE CARLO SIMULATION

The use of the monotonicity property is as follows. A. The Zero-Order Sugeno FIS Model

The fuzzy production rules for an \( n \)-input FIS model, where \( n > 0 \), can be represented as follows. Let \( x_{i_1}, x_{i_2}, ..., x_{i_n} \):

\[ \text{IF } (x_i \text{ is } A_{i_1}^{\mu_1}) \text{ AND } (x_j \text{ is } A_{i_2}^{\mu_2}) \text{ ... AND } (x_k \text{ is } A_{i_n}^{\mu_n}) \text{ THEN } (y \text{ is } B_{j_1}^{\mu_1} \text{ ... } B_{j_n}^{\mu_n}) \]

1 \leq i, j \leq M_i.
The AND operator in the rule antecedent part is the product function. For the \( x \) domain, its Gaussian MFs are \( \mu^1_1(x; c^1_1, \sigma^1_1) \), \( \mu^2_1(x; c^2_1, \sigma^2_1) \), \( \ldots \), \( \mu^M_1(x; c^M_1, \sigma^M_1) \) (for linguistic terms \( A^1_1, A^2_1, \ldots, A^M_1 \) respectively). The upper and lower limits for the universe of discourse of \( x \) are \( x_i \) and \( x_l \), respectively. The output is obtained by using the weighted average of a representative value, \( b_{i_1,i_2,\ldots,i_N} \), with respect to its compatibility grade, as in (4).

\[
y = \frac{\sum_{j_1=1}^{n_1} \mu^1_{j_1}(x_1) \sum_{j_2=1}^{n_2} \mu^2_{j_2}(x_2) \ldots \sum_{j_M=1}^{n_M} \mu^M_{j_M}(x_M) \times b_{i_1,i_2,\ldots,i_N} \times \prod_{j=1}^{M} (\frac{1}{\sigma_j^M} - \frac{1}{\sigma_j^M})}{\sum_{j_1=1}^{n_1} \sum_{j_2=1}^{n_2} \sum_{j_M=1}^{n_M} (\frac{1}{\sigma_j^M} - \frac{1}{\sigma_j^M})}
\]

where \( b_{i_1,i_2,\ldots,i_N} \) is the representative value of \( B_{i_1,i_2,\ldots,i_N} \), i.e., \( b_{i_1,i_2,\ldots,i_N} = \text{Rep}(B_{i_1,i_2,\ldots,i_N}) \). This value represents the overall location of the MF. It can be obtained by defuzzification, or be represented by the point whereby the membership value is 1.

### B. The Sufficient Conditions

The first derivative of an FIS model, as in (4), returns a weighted addition series. The sufficient conditions assume that all the components in the weighted addition series are always greater than or equal to zero, or less than or equal to zero. Two conditions can be derived [1], as follows.

**Condition 1.** At the rule antecedent part, \((d \mu_i^j(x)/dx)/\mu_i^j(x) \geq (d \mu_i^j(x)/dx)/\mu_i^j(x)\), where \( p > q \). Note that \((d \mu_i^j(x)/dx)/\mu_i^j(x)\) is the ratio between the rate of change in the membership degree and the membership degree itself. The derivative of a Gaussian MF with respect to \( x \) is \( G(x) = -(x-c)/\sigma G(x) \). Note that \((d \mu_i^j(x)/dx)/\mu_i^j(x)\) for a Gaussian MF, i.e., \((G(x)/G(x))\), returns a linear function, i.e. \( E(x) = G(x)/G(x) = -(1/\sigma G(x)) + (c/\sigma^2) \).

**Condition 2.** At the rule consequent part, \( b_{i_1,i_2,\ldots,i_N} \) is the ratio between the rate of change in the output and the output itself, \( dy/dx \leq 0 \), respectively. This condition suggests that a monotonic rule base is required.

### C. Monte Carlo Simulation

Monte Carlo simulation provides approximate solutions by performing statistical sampling experiments [15]. It can be used to study the monotonicity property of an FIS model, particularly the relationship of the developed mathematical conditions, the exact fulfillment of the monotonicity property, and the non-monotonicity property. Fig. 1 illustrates the Monte Carlo method. FIS models, parameterized by \( \theta \), are randomly generated. The relationship of the FIS model, its monotonicity property and the developed mathematical conditions (e.g., the sufficient conditions for the Sugeno FIS model, i.e., condition 1 and condition 2) are studied.

Table 1 summarizes the experimental results for a single-input \( (n = 1) \) FIS model with \( M_1 \) fuzzy rules (i.e., \( \theta \rightarrow \mu_i(x; c_i, \sigma_i) \rightarrow b_i \) where \( j = 1,2,\ldots,M_1 \)) with Gaussian MFs. The parameters associated with \( \theta \) are \( c_i, \sigma_i \) and \( b_i \). We examine the fulfillment of the monotonicity property (via the monotonicity index) and the sufficient conditions for a randomly generated FIS model. Details of the experiment are summarized in Fig. 2.

![Figure 1. Pseudo-code of the Monte Carlo simulation](image1)

![Figure 2. Parameters setting for the Monte Carlo simulation](image2)

The numbers of FIS models that fulfil the monotonicity index and/or the sufficient condition are shown in Table 1. Columns Monotonicity Index = 1 and Monotonicity Index = 1 show the numbers of randomly generate FIS models that fail and pass the monotonicity test, respectively. Columns "Conditions 1 and 2 not fulfilled" and "Conditions 1 and 2 fulfilled" show the number of randomly generate FIS models that fulfil and violate the sufficient conditions, respectively. As an example, for \( M_1 = 3 \), 8868 FIS models fail the monotonicity test, and none of them pass the sufficient conditions test too. There are a total of 1132 (1015 + 117) FIS models that pass the monotonicity test, with only 117 pass the sufficient conditions test.

<table>
<thead>
<tr>
<th>Number of membership functions, ( M_1 )</th>
<th>Monotonicity index = 1</th>
<th>Monotonicity index = 1</th>
</tr>
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<td>Conditions 1 and 2 fulfills.</td>
<td>Conditions 1 and 2 not fulfilled.</td>
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</tr>
<tr>
<td>10</td>
<td>9725</td>
<td>0</td>
</tr>
</tbody>
</table>

### D. Discussions

From the experimental results in Table 1, a few observations can be made, as follows.

(a) It is possible to represent the relationships among FIS (i.e., \( FIS \)), FIS with the sufficient conditions fulfilled (i.e., \( m_{FIS,wsc} \)) and FIS with fulfillment of the monotonicity property (evaluated by using the monotonicity index, i.e., \( M_1 \)), as in Fig. 3. Note that \( m_{FIS,wsc} \) is a subset of \( m_{FIS} \), i.e., \( m_{FIS,wsc} \in m_{FIS} \).

(b) Region \( m_{FIS} \) or \( m_{FIS,wsc} \) is relatively large, as compared with region \( m_{FIS,wsc} \). The exploitation of the sufficient conditions allows investigations on the \( m_{FIS,wsc} \) region only. It would be useful to work on the \( m_{FIS} \) region, which is an open area for research.
In general, an optimization-based FIS model can be constructed by using a search procedure to search for a set of \( \theta = \theta^* \), such that the resulting FIS model, \( y = f(\hat{x}; \theta^*) \), is able to describe the model appropriately \([13-16]\), e.g., with the lowest mean square error (with a good precision), with good interpretability, and etc. Usually, the FIS model needs to be optimized by using an objective function, \( g(\theta) \). Various optimization methods exist, e.g., GA-based, derivative-based optimization, as well as memetic-based approaches. Among them, the GA is a useful method for constructing an optimization-based FIS model \([13]\).

It is possible to classify the role of the GA in optimization-based FIS models into two categories, i.e., learning and tuning \([13]\). The former focuses on the learning of a knowledge base and the parameters for an FIS model. The latter focuses on tuning of the FIS parameters without changing the original FIS knowledge base, with the aim to achieve better performances. The monotonicity index can be formulated as a constraint, i.e., \( h(\hat{x}; \theta) = 1 \), for the optimization problem. In this paper, we suggest to search for a set of \( \theta = \theta^* \), where its \( h(\hat{x}; \theta) = \text{Monotonicity Index} (y, x_i) = 1 \), if \( y \) follows the monotonicity relationship with \( x_i \).

A data-driven FIS model is equipped with the capability to learn from a set of data samples for a problem under scrutiny. Consider a system identification problem that attempts to determine a mathematical model (which can be an FIS model) for an unknown system by observing \( m \) desired input-output pairs \( (\hat{x}^k, y^k), k = 1, 2, 3, ..., m \). According to \([16]\), a system identification problem can be summarized into 4 steps:

1. Specify and parameterize a class of mathematical models, e.g., an FIS model, i.e., \( y = f(\hat{x}; \theta) \), that represents the system to be identified;
2. Perform parameter identification to choose a set of \( \theta = \theta^* \) that best fits the data set, \( (\hat{x}^k, y^k) \);
3. Conduct a validation test to see if the model has been identified correctly for the problem under scrutiny in accordance with the test data set;
4. Terminate the procedure once the validation test is satisfactory. Otherwise, go back to step (1).

The focus of this study is to optimize an FIS model (Step (2)), \( y = f(\hat{x}; \theta) \) to best fit a training data set. This can be implemented by formulating the data-driven FIS model as a search of a set of \( \theta \) such that \( g(\theta) \), an objective or error function, is minimum subject to a constraint, \( h(\hat{x}) \), as follows.

\[
g(\theta) = \sum_{k=1}^{m} \left( y^k - f(\hat{x}^k; \theta) \right)
\]

\[
h(\hat{x}; \theta) = \text{Monotonicity Index} (y, x_i) = 1
\]

\[
k = \begin{cases} 0 & \text{if } h(\hat{x}; \theta) = 1 \\ 1 & \text{if } h(\hat{x}; \theta) = 0 \end{cases}
\]

The GA can be used to solve this problem. In this paper, the objective function and the constraint is combined into a single objective function, as follows.

\[
\text{obj} = g(\theta) + w \times k
\]

where \( w \) is a constant. Fig. 4 illustrates the procedure of the proposed GA-based FIS model that preserves the monotonicity property.

### A Simulated Problem

A simple quadratic function, i.e., \( y = x^2 \), is considered. We attempt to model the quadratic function using an FIS model, with \( x \) ranges from 0 to 1. It is an example of a monotonic function. Two data sets, as shown in Table II, are used for simulation purposes. In the third column (Data set \( I \)), it is assumed that there is no error in the data samples, which are evenly distributed. In the fourth column (Data set \( 2 \)), it is assumed that errors exist in the data samples (e.g., data sample 2), and some data samples are missing (e.g., data points 3 to 7). As an example, notice that for data sample 2, \( y = 1 \) when \( x = 1.00 \). This is an outlier that causes the data set to be non-monotonic.

### Table II. Data Used

<table>
<thead>
<tr>
<th>Data points</th>
<th>( x )</th>
<th>( y ) (Data set 1)</th>
<th>( y ) (Data set 2)</th>
</tr>
</thead>
<tbody>
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</table>
According to [12], there are three options to handle non-monotonic data samples: (1) keep the data samples as they are; (2) identify and remove the data samples; (3) identify and re-label the data samples. In this paper, we adopt the first option, and the monotonicity property is viewed as a useful (additional) information to overcome the over-fitting problem and to rectify the error.

B. The Zero-Order Sugeno FIS Model

A zero-order Sugeno FIS with Gaussian MFs is considered. The $x$ domain is represented by $j$ MFs with $j$ fuzzy rules. A fuzzy rule can be parameterized as $\mu_l(x; c_l, \sigma_l) \rightarrow b_l$, where $l = 1, 2, 3, ..., j$. Note that $c_l, \sigma_l$, and $b_l$, are elements of $\theta$. The GA is used to search for the minimum value of $g(\theta)$, subject to $h(x; \theta) = 1$, where $w$ is 10.

Figures 5(a) and 5(b) show the objective function of the GA-based Sugeno FIS model with the monotonicity index and the plot of $y$ versus $x$ for the GA-based Sugeno FIS model, respectively, with data set 1 from Table II for 150 iterations. From Figure 5(b), a monotonic function with objective value=0.0059, is obtained.

Figures 6(a) and 6(b) show the objective function of the GA-based Sugeno FIS model with the monotonicity index and the plot of $y$ versus $x$ for the best GA-based Sugeno FIS model with data set 2, respectively. In Figure 6(b), a monotonic function with an objective function of 0.4161 is obtained. In Figure 7, the plot of $y$ versus $x$ for the GA-based Sugeno FIS model, without the use of the monotonicity index, is presented. An objective value of 0.0087 is obtained. Even through the objective value for the GA-based Sugeno FIS model without the monotonicity index is better than that with the monotonicity index, it is not able to preserve the monotonicity property, as illustrated by the non-monotonic curve in Figure 7. With the monotonicity index, we obtain a monotonicity index of 0.5160.

C. The Tsukamoto FIS Model

Reasoning in a Tsukamoto FIS model can be summarized as in Figure 8. Again, a Tsukamoto FIS model with $j$ fuzzy rules is considered. Each fuzzy rule is parameterized as $\mu_l(x; c_l, \sigma_l) \rightarrow f(\gamma_l)$, where $l = 1, 2, 3, ..., j$. The GA is used to search for $c_l, \sigma_l, \gamma_l^0$, and $\gamma_l^1$, which are elements of $\theta$, such that $g(\theta)$ is minimum, subject to $h(x; \theta) = 1$. With our proposed method, it is possible to construct a monotonic Tsukamoto FIS model, without having to rely on certain specific mathematical condition(s).

Figures 9(a) and 9(b) show the objective function of the GA with the monotonicity index and the plot of $y$ versus $x$ for the GA-based Tsukamoto FIS model with the monotonicity index, respectively, with data set 1, for 150 iterations. In
Figure 9(b), a monotonic function with objective value=0.0196 is shown.

Figure 9. (a) The objective function of the GA-based Tsukamoto FIS model with the monotonicity index using data set 1, (b) Plot of y versus x for the best GA-based Tsukamoto FIS model with the monotonicity index (objective value= 0.0196, monotonicity index=1.00)

Figures 10(a) and 10(b) as well as Figure 11 illustrate the experimental results with data set 2 for 150 GA iterations. Figures 10(a) and 10(b) show the objective function of the GA-based Tsukamoto FIS model with the monotonicity index and the plot of y versus x for the best GA-based Tsukamoto FIS model, respectively. In Figure 10(b), a monotonic function with an objective function of 0.3240 is obtained. In Figure 11, the plot of y versus x for the GA-based Tsukamoto FIS model, without the monotonicity index, is presented. An objective value of 0.0092 is obtained. Even though the objective value for the GA-based Tsukamoto FIS model without the monotonicity index is better than that with the monotonicity index, it is not able to preserve the monotonicity property (i.e., monotonicity index=0.60).

Table III. DATA USED

<table>
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A. The Zero-Order Sugeno FIS Model

Figures 12(a) and 12(b) show the objective function of the GA-based Sugeno FIS model with the monotonicity index and the plot of y versus x for the best GA-based Sugeno FIS model, respectively, with data set 3 for 150 iterations. From Figure 12(b), a function with the local monotonicity property satisfied with objective value= 0.0490 is obtained.

Figures 13(a), 13(b), and 14 illustrate the experimental results with data set 4 for 150 GA iterations. Figures 13(a) and 13(b) show the objective function of the GA-based Sugeno FIS model with the monotonicity index and the plot of y versus x for the best GA-based Sugeno FIS model, respectively. The best GA-based Sugeno FIS model with the...
monotonicity index returned an objective value of 1.1443, as shown in Figure 13(b). The function fulfils the local monotonicity property.

In Figure 14, the plot of y with respect to x for the best GA-based Sugeno FIS model without the monotonicity index is presented. An objective value of 0.4806 is obtained. Even through the objective value for the GA-based Sugeno FIS model without the monotonicity index is better than that with the monotonicity index, it cannot preserve the local monotonicity property, as shown in Figure 14.

The Tsukamoto FIS Model

Figures 15(a) and 15(b) show the objective function of the GA-based Tsukamoto FIS model with the monotonicity index and the plot of y versus x for the best GA-based Tsukamoto FIS model, respectively, with data set 3 for 150 iterations. From Figure 15(b), a function with the local monotonicity property satisfied with objective value = 0.00047521 is depicted.

Figures 16(a), 16(b) and 17 illustrate the experimental results with data set 4 for 150 GA iterations. Figures 16(a) and 16(b) show the objective function of the GA-based Tsukamoto FIS model with the monotonicity index and the plot of y versus x for the best GA-based Tsukamoto FIS model, respectively. As shown in Figure 16(b), a function with an objective value of 0.9074 is obtained. The function fulfils the local monotonicity property. In Figure 17, the plot of y versus x for the best GA-based Tsukamoto FIS model without the monotonicity index is presented. An objective value of 0.2447 is obtained. Even through the objective value for the GA-based Tsukamoto FIS model without the monotonicity index is better than that with the monotonicity index, it cannot preserve the local monotonicity property, as shown by the non-local monotonic curve in Figure 17.
VI. SUMMARY

In this paper, we have introduced the use of a monotonicity index for tackling the monotone fuzzy modeling problem. The focus of this paper is on data-driven GA-based FIS models i.e., the use of the GA to construct an FIS model based on a set of data. The monotonicity (and local monotonicity) property is exploited as an additional qualitative information, or prior knowledge, to mitigate the over-fitting phenomenon. The monotonicity index is used as a constraint to guide the optimization process.

We have demonstrated the importance of the monotonicity index for FIS modeling using the Monte Carlo simulation. The outcomes have shown that the probability of a randomly generated monotonic Sugeno FIS model to fulfill the sufficient conditions is relatively small, and decreases drastically as the number of MFs increases. In short, a relatively large region which has not been described by the sufficient conditions exists. We have further demonstrated the use of the monotonicity index in data driven GA-based FIS model building. Two FIS structures, i.e., the zero order Sugeno FIS model and the Tsukamoto FIS model, for undertaking system identification problems with the monotonicity and local monotonicity properties have been studied. Simulated data sets have been used, and promising results have been obtained. We have shown that with the use of monotonicity index as a constraint, better results that can preserve the local monotonicity property can be obtained. In summary, this study has shown that the monotonicity index is useful, and can be used to construct a monotonicity-preserving FIS model, even if the mathematical conditions for preserving the monotonicity property are unknown. To the best of our knowledge, investigations into the local monotonicity property in FIS modeling is new, even though its importance has been highlighted in the study of signal processing [17]. Hence, this constitutes a new area for undertaking the monotone fuzzy modeling problem.

As future work, we will extend the proposed approach to multi-objective optimization, in which the monotonicity index is one of the objectives instead of a constraint. Real-world case studies will also be conducted to vindicate the usefulness of the proposed approach in building FIS models for undertaking problems that require the monotonicity property.

REFERENCES