Abstract—The polynomial chaos-based stochastic Galerkin method is implemented in a magnetized plasma finite-difference time-domain (FDTD) algorithm to characterize the uncertainty characteristics of electromagnetic wave propagation in the ionosphere. This new algorithm efficiently calculates in a single simulation not only the mean electromagnetic field values, but also their standard deviation as caused by the variability or uncertainty of the content of the ionosphere. This approach represents a paradigm shift in our ability to analyze realistic, complex wave propagation in the ionosphere. The algorithm is validated by comparing with Monte Carlo results.

I. INTRODUCTION

The finite-difference time-domain (FDTD) method [1], [2] has been applied to electromagnetic wave propagation in the Earth-ionosphere waveguide for well over a decade (e.g. [3], [4], [5], [6], [7], [8]). In recent years, FDTD anisotropic magnetized ionospheric plasma models have been incorporated into global FDTD models to extend their capabilities. Specifically, a fully three-dimensional (3-D) Cartesian plasma model developed in [9] was applied to the 3-D FDTD latitude-longitude global spherical grid [10] of [5]. By accounting for magnetized ionospheric plasma physics, [9] was the first global FDTD model to include the calculation of all important ionospheric effects on signals, including absorption, refraction, phase and group delay, frequency shift, polarization, and Faraday rotation.

All of the FDTD plasma models to date (both the simplified isotropic conductivity profile plasma and more advanced magnetized plasma versions) use only average (mean) values of the constitutive parameters of the content of the ionosphere and then solve for expected (mean) electric and magnetic fields without considering uncertainties (e.g. [11]). Communications, surveillance, and navigation capabilities rely heavily on accurate knowledge of electromagnetic (EM) signal propagation through and reflected by the Earth’s ionosphere. It is crucial for the performance of these systems to include not just the general (average) structure of the ionosphere, but also its variability (or uncertainty). For example, the variability of the ionosphere strongly affects trans-ionospheric radio propagation. The irregularities in the electron density distribution can cause phase and amplitude scintillation. Therefore, uncertainty analysis becomes an important factor to be considered in ionosphere electrodynamics modeling.

The ionosphere undergoes variations due to the influence of solar and geomagnetic events, lightning, as well as other sources. Recent investigations into the temporal and spatial variations of the ionosphere have improved our understanding of the structure of the ionosphere. This variability creates significant complexities for EM propagation problems, requiring them to be solved using a deterministic formulation. A useful approach to complex ionospheric propagation problems is to consider them as random medium problems. The Monte Carlo method is a widely-used brute force technique for evaluating random medium problems via multiple realizations. However, this method has a well-known drawback of being computationally expensive, preventing it from being a feasible approach when considering large propagation distances, especially in 3-D.

In order to overcome the limitations of the Monte Carlo approach, [12] successfully extended the single-realization, Maxwell’s equations stochastic FDTD (S-FDTD) methodology developed by Smith and Furse and applied to biomedical applications [13] to the fully 3-D anisotropic magnetized plasma algorithm of [9]. Although the advantage of the resulting S-FDTD plasma algorithm is that it requires only about twice as much computer simulation time and memory regardless of the number of random input parameters, its limitation is that the accuracy relies on the best estimate approximation for the cross correlation coefficients. Moreover, because this method is based on the truncation of Taylor series, wherein the mean value is only first-order accurate and the standard deviation value is second-order accurate. This limitation in accuracy is unfortunate since the basic FDTD algorithm is second-order accurate. Thus, S-FDTD may be insufficient for some applications.

This paper presents a promising alternative approach to the S-FDTD magnetized plasma algorithm [12]. It utilizes the spectral expansion method or polynomial chaos expansion (PCE) [14], [15] to represent the stochastic processes and resulting variability of the EM wave propagation in the magnetized cold plasma medium. PCE is based on the expansion of the output quantities of interest into a sum of a limited number of orthogonal polynomials. The PCE method is robust and versatile method since it can be applied even to non-Gaussian problems, and depending on the requirements of individual propagation problems, the accuracy is improved by simply
increasing the order of orthogonal polynomials.

The ultimate objective is to develop an efficient stochastic FDTD-based algorithm that is well-suited for the especially large uncertainty quantification exhibited by the ionosphere (up to 100% or more), and then use it to understand the variability of the EM wave propagation through that complex ionosphere. PCE appears to be a very promising approach to meet this objective.

II. FORMULATION

A. Boris algorithm

The magnetized (anisotropic) cold plasma governing equations are cast in terms of Maxwell’s equations coupled to current equations derived from the Lorentz equation of motion [9]. The resulting whole governing equation set is given by:

\[ \nabla \times E = -\mu_0 \frac{\partial H}{\partial t} \]  
\[ \nabla \times H = \epsilon_0 \frac{\partial E}{\partial t} + J_c \]  
\[ \frac{\partial J_c}{\partial t} + v_e J_c = \epsilon_0 \omega_p^2 E + \omega_C e J_c \]  

Here \( v_e, J_c, \omega_C e \) and \( \omega_p e \) are the collision frequency, the current density, the cyclotron frequency and the plasma frequency of electrons, respectively. The plasma frequency is a function of the electron density \( n_e \) given by,

\[ \omega_p e = \left( \frac{q_e^2 n_e}{\epsilon_0 m_e} \right)^{1/2} \]  

The cyclotron frequency of the electrons is given by \( \omega_C e = \frac{|q_e| B}{m_e} \) where \( B \) is the applied magnetic field. The equations (2) and (3) imply that this scheme is implicit. As a result, the governing stochastic equations take the form of a large, complex matrix [9]. Such a complex physical model prevents us from applying the PCE method because it is not possible to obtain the derivation of the explicit equations for the PCE coefficients. Therefore, a new methodology for solving electromagnetic wave propagation in magnetized plasma is needed in order to make it feasible to apply the PCE method.

For the general case, all three input parameters \( (v_e, n_e \) and \( B \) in equation (3) can be considered random parameters. Here we will consider the simplest case: a collision-less plasma \( (v_e = 0) \) with \( n_e \) as the only random input parameter.

Recently, a more efficient 3-D FDTD magnetized plasma algorithm was developed [16] by employing the Boris algorithm originally applied to particle-in-cell plasma modeling [17], [18]. In this case, the Lorentz equation is solved explicitly and is easily incorporated into the traditional FDTD Maxwell’s equations. The Boris scheme using two auxiliary current density vectors is as follows:

\[ J_c^+ = J_c^{e^{-1/2}} + \frac{\Delta t \epsilon_0 \omega_p^2 E^n}{2} \]  
\[ J_c^- = J_c^{e^{+1/2}} - \frac{\Delta t \epsilon_0 \omega_p^2 E^n}{2} \]  

Then equation (3) is simplified to:

\[ \frac{J_c^+ - J_c^-}{2} = -\omega_C e \times \left( \frac{J_c^+ + J_c^-}{2} \right) \]  

Here, \( J_c^+ \) is calculate from equation (7) through a rotation of \( J_c^- \) at angle of ration \( \theta \) as shown in Fig.1 and following equation [16]:

\[ J_c^+ = J_c^- + [J_c^- + (J_c^- \times t)] \times s \]  

where,

\[ t = -\omega_C e \frac{\tan \theta}{\omega_C e \sin \theta}; \quad s = \frac{\omega_C e}{\omega_C e} \tan \theta; \quad \theta = \tan^{-1} \frac{|\omega_C e| \Delta t}{2} \]

Now, all the governing equations are solved explicitly and may be derived in discrete form. For example, considering the \( x \)-components, the resulting equations are shown as equations (9)-(11). Note that in this algorithm, the current density vector components are spatially collocated with an electric field component, but in the time domain, the current density vectors are calculated at the same time as the magnetic field components. Therefore, the calculation of the electric field follows the calculation of the magnetic field and current density in a leapfrog scheme.

In order to increase the computational efficiency, all current density vectors are calculated at the same grid point as an electric field component. Then, the average of the four neighboring electric field components (or current densities) are used as needed in the updating equations. Fig.2 illustrates the positions of the electromagnetic field and current density components in the Yee cell in which all the current density components are solved at the position of \( E_x \).

B. Polynomial chaos expansion

Ionosphere electron densities vary in a complex manner as a function of location and time. Thus, we consider the electron density as a random variable with its own statistical variation. The electron density is assumed to follow a normal (Gaussian) distribution having mean value of \( \mu_{n_e} \) and standard deviation value of \( \sigma_{n_e} \). The variability in the electron density causes variability in the EM fields and current density \( (E, H \) and \( J_c) \), which will be treated as output random variables.
Next, in order to implement the polynomial chaos method, the uncertain fields are expanded. As an example, the expansion of only the $x$-component is shown as follows:

\begin{align}
H_x^{n+1/2} &= H_x^{n-1/2} + \frac{\Delta t}{\mu_0} \left[ \frac{E_y^{n+1/2}}{\Delta z} - \frac{E_y^{n+1/2}}{\Delta y} \right] \\
J_{ex}^{n+1/2} &= J_{ex}^{n-1/2} + \frac{\Delta t}{\mu_0} \left[ n_e E_x^{n+1/2} - \frac{\sin \theta}{\omega_C e} J_{ey}^{n+1/2} + \frac{\Delta t^2 \sin \theta}{\mu_0} \left( \frac{E_x^{n+1/2} \omega_C e - E_x^{n+1/2} \omega_C e}{} \right) \right] \\
E_x^{n+1/2} &= E_x^{n+1/2} + \frac{\Delta t}{\epsilon_0} \left[ \frac{H_z^{n+1/2}}{\Delta z} - \frac{H_z^{n+1/2}}{\Delta y} \right]
\end{align}

where $h_{x}, j_{ex}, e_{x}^{a}$ are the weighting coefficients, $\xi$ is the standard normal random variable with zero mean and unity standard deviation, and $\psi$ are Hermite polynomials. The choice of the orthogonal basis functions depend on the distribution of the random variables being considered. In this case, Hermite polynomials corresponding to a normal distribution.

The number of terms is given by:

\begin{align}
P + 1 &= \frac{(n + d)!}{n!d!}
\end{align}

where $d$ is the order of the highest order Hermite polynomial used in the expansion and $n$ is the number of random variables.

The electron density $n_e$ can be related to $\xi$ through its mean $\langle n_e \rangle$ and standard deviation $\sigma_{n_e}$ as:

\begin{align}
n_e(\xi) &= \mu_{n_e} + \sigma_{n_e} \xi
\end{align}

Or in polynomial form:

\begin{align}
n_e(\xi) &= \sum_{b=0}^{P} n_e^b \psi_b(\xi)
\end{align}

where,

\begin{align}
n_e^0 &= \mu_{n_e}, \quad n_e^1 = \sigma_{n_e}, \quad \text{and} \quad n_e^b = 0 \quad \text{with} \quad b > 1
\end{align}

There are two approaches for evaluating the weighting coefficients $h_{x}^{a}, j_{ex}^{a}, e_{x}^{a}$: the Galerkin method and the collocation method. Here, the Galerkin method is chosen over the collocation method because modeling of electromagnetic wave propagation in magnetized plasma is a large-scale problem where a single deterministic computation is already time consuming. The expansions of $H_x, J_{ex}$ and $E_x$ are substituted into (9)-(11). Then, the Galerkin method is applied by taking the inner products of the expansion equations with the test function $\psi_a(\xi)$, where $c = 0, ..., P$. Next the following orthogonality condition is used:

\begin{align}
\langle \psi_a(\xi), \psi_b(\xi) \rangle = \langle \psi_a(\xi) \rangle \delta_{a,b}
\end{align}

where $\delta_{a,b}$ is the Kronecker delta function ($\delta_{a,b} = 0$ if $a \neq b$ and $\delta_{a,b} = 1$ if $a = b$). This orthogonality reduces
the expansion equations to a set of \((P + 1)\) uncoupled and deterministic equations.

Once the coefficients \(c_{0}, b_{a}, d_{a}(u = x, y, z)\) are found, the mean and variance of the output fields can be obtained. For example, for the \(E_{x}\) field:

\[
\mu \left[ E_{x, i+1/2,j,k} (\xi) \right] = e_{x, i+1/2,j,k}^{0} (18)
\]

\[
\sigma^{2} \left[ E_{x, i+1/2,j,k} (\xi) \right] = \sum_{a=1}^{P} (e_{x, i+1/2,j,k}^{a})^{2} \langle \phi_{a}^{2} \rangle (19)
\]

III. NUMERICAL EXAMPLE

The performance of the polynomial chaos-based stochastic Galerkin approach for uncertainty EM propagation modeling in fully 3-D FDTD magnetized cold plasma described in Section II is evaluated by running a similar validation test as for the FDTD plasma model of [9]. An \(x\)-polarized, \(z\)-directed Gaussian-pulsed plane wave is implemented. Assuming there is no uncertainty in the source, the weighting coefficients of the source are implemented as:

\[
e_{x,source}^{a} = \begin{cases} 
\exp \left[ \frac{-\left( \xi - 5\Delta t \right)^{2}}{2(\Delta t)^{2}} \right] & \text{if } a = 0 \\
0 & \text{if } a > 0
\end{cases}
\]

The lattice space increments in each Cartesian direction of the grid are \(\Delta x = \Delta y = \Delta z = 1mm\), the time step \(\Delta t = \Delta z / (c \times 0.55)\). An magnetic field \(B = 0.06T\) is applied to the plasma (a large value so that we may observe an effect of the plasma over a short distance for validation purposes).

For validation, Monte Carlo simulations are used to predict the exact mean and standard deviation of the fields. First, the input electron densities \(n_{e}\) for each simulation are generated in a random manner with a normal distribution given by mean \(\mu_{n_{e}} = 1.0 \times 10^{18} m^{-3}\) and standard deviation \(\sigma \{n_{e}\} = 2.0 \times 10^{17} m^{-3}\) (equivalently \%\sigma \{n_{e}\} = 20\%). All of the simulation responses are collected and analyzed to obtain their statistical properties (mean and standard deviation values). Then, using polynomial chaos-based FDTD, four separate simulation cases are run using the truncation of polynomial chaos expansion of order \(d \in \{1, 2, 3, 4\}\).

Fig. 3 shows the time-domain waveform of the mean output electric field \(E_{x}\) as calculated by the polynomial chaos FDTD approach. The corresponding standard deviation as recorded 10-cells away from the source in the \(z\)-direction is shown in Fig. 4. Monte Carlo FDTD results are also shown for comparison. In these results, truncating the polynomial chaos expansion at a higher order of \(d\) yields a better uncertainty prediction of the electromagnetic field, as expected. The results show that an order of at least \(d = 2\) is required for the mean prediction and an order \(d = 4\) is required for the polynomial chaos standard deviation prediction to agree with Monte Carlo results. This finding is analogous to the Monte Carlo method wherein its accuracy depends on the number of Monte Carlo simulations (i.e. increasing the number of Monte Carlo simulations helps converge better to the exact results). However, for the polynomial chaos-based Galerkin approach, one single simulation is sufficient to obtain a complete statistical characterization, and the accuracy depends on the order of polynomial chaos considered (e.g. increasing the order of the expansion improves the accuracy of the results).

IV. CONCLUSION

This paper introduced a novel approach for using the polynomial chaos-based Galerkin approach for uncertainty quantification of EM wave propagation in a 3-D FDTD magnetized cold plasma model. This model is derived from Maxwell’s equations coupled to current equations derived from the Lorentz equation of motion. The statistical characteristics (mean and standard deviation) of the EM wave propagation are studied under the effect of an univariate random medium parameter, i.e the electron density. The results demonstrated very good agreement with the Monte Carlo result while offering an exceptional improvement in simulation time. The
polynomial chaos approach thus shows promise for efficiently generating accurate predictions of the statistical behavior of a realistic, complex wave propagation in the ionosphere. It may therefore serve as an important tool for reliably estimating the uncertainty of EM ionospheric propagation studies, especially for large 3-D plasma scenarios wherein Monte Carlo simulations would be impractical to run. Future research will involve investigations of more complex scenarios by considering the collisional regime and involving multivariate uncertain inputs (collision frequency, cyclotron frequency and electron/ion densities).

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