Electromagnetic Wave Scattering by a Small Impedance Body of an Arbitrary Shape

Alexander Ramm
Kansas State University
Manhattan, KS 66506, USA
E-mail: ramm@math.ksu.edu

Abstract—It is proved that electromagnetic wave scattering problem for a small impedance body $D$ of an arbitrary shape is uniquely solvable and the scattered field can be found in the form $e = \nabla \times \int_S g(x,t)J(t)dt$, where $S$ is the smooth surface of the small body, $J$ is a tangential field on $S$, and $g(x,t) := \frac{e^{i\delta(x-t)}}{4\pi|x-t|}$. The full field is $E = E_0 + e$, where $E_0$ is the incident field. The scattering amplitude is

$$A(\alpha, \beta, k) = -\frac{\zeta|S|}{4\pi} \sqrt{\frac{\tau}{\mu}} [\beta, \tau \nabla \times E_0(O)],$$

where the origin $O$ is assumed to be inside $D$, $\beta := \frac{r}{|r|}, |r| \to \infty$, $\tau := \tau_{pq} := \delta_{pq} - \frac{1}{|S|} \int_S N_p(t)N_q(t)dt$, where $dt$ is the element of the surface area.

Let the incident wave be $E_0 = \mathcal{E} e^{i\alpha \cdot x - \omega t}$, where $\mathcal{E} \cdot \alpha = 0$, $\alpha \in S^2$, $S^2$ is the unit sphere in $\mathbb{R}^3$, $\mathcal{E}$ is a constant vector, $k = \text{const} > 0$ is the wave number, $k = \frac{2\pi}{\lambda}$, $\lambda$ is the wavelength. Let $D' = \mathbb{R}^3 \setminus D$ and $a = \frac{1}{2}\text{diam}D$. Assume that $ka \ll 1$, that is, the body is small.

Consider the Maxwell’s equations:

$$\nabla \times E = i\omega \mu H, \quad \nabla \times H = -i\omega \epsilon E \quad \text{in } D',$$

(1)

$\epsilon$ and $\mu$ are dielectric and magnetic constants in $D'$, $\omega > 0$ is the frequency. Let us seek a solution to equations (1) of the form

$$E = E_0 + e, \quad H = \frac{\nabla \times E}{i\omega \mu} := H_0 + h,$$

(2)

where $e$ is the scattered field, $e$ satisfies the radiation condition:

$$e_r - ike = o \left( \frac{1}{r} \right), \quad r := |x| \to \infty, \quad e_r := \frac{\partial e}{\partial r},$$

(3)

and the impedance boundary condition on $S$:

$$[N[e,N]] - \zeta[N,h] = -f,$$

(4)

$$f := [N,[E_0,N]] - \zeta[N,H_0].$$

Here $[A,B] := A \times B$ is the vector product of two vectors and $(A,B) := A \cdot B$ is their scalar product.

Electromagnetic (EM) wave scattering problem consists of finding the solution to problem (1) - (4). We look for the scattered field $e$ of the form

$$e = \nabla \times \int_S g(x,t)J(t)dt, \quad g(x,t) = \frac{e^{i\delta|x-t|}}{4\pi|x-t|},$$

(5)

where $J$ is a tangential to $S$ field.

We prove that if $\text{Re} \zeta \geq 0$ and $D$ is sufficiently small, then problem (1) - (4) has a solution of the form (5) and this solution is unique. The smallness of $S$ is described by the inequality $ka \ll 1$, where $k = \omega^2 \epsilon \mu$.

We derive a closed form expression for $e$:

$$e = [\nabla_2 g(x,x_1), Q],$$

(6)

$$Q := \int_S J(t)dt = -\frac{\zeta|S|}{i\omega \mu} \tau(\nabla \times E_0)(x_1),$$

where $x_1 \in D$ is an arbitrary point, and

$$\tau_{pq} = \delta_{pq} - \frac{1}{|S|} \int_S N_p(t)N_q(t)dt,$$

(7)

Formula (7) is valid for $|x - x_1| \gg a$.

A simple calculation, based on formulas (6), yields the following formula for the scattering amplitude $A(\beta, \alpha, k)$:

$$A(\beta, \alpha, k) = -\frac{\zeta|S|}{4\pi} \left( \frac{\epsilon}{\mu} \right)^{1/2} [\beta, \tau \nabla \times E_0(x_1)],$$

(8)

where $\beta := \frac{x-x_1}{|x-x_1|}, |x - x_1| \to \infty.$
The theory of wave scattering by small bodies (particles) was originated by Lord Rayleigh in 1871, (see [6]), who understood that the main term in the scattered field $e$ is the dipole radiation. He did not give formulas for calculating this radiation for small bodies of an arbitrary shape. This was done by the author in [1]- [4]. In [5] the theory was developed to include scalar wave scattering for various boundary conditions (the Dirichlet, Neumann, impedance, and interface boundary conditions) and the problem of wave scattering by many small bodies of arbitrary shapes.

In this paper mathematical foundations of the electromagnetic (EM) wave scattering theory for small impedance particles of an arbitrary shape are given.

In section 2 the results are formulated and proofs are given.

II. FORMULATION OF THE RESULTS AND PROOFS

The first result is known (see [5], p. 81):

**Lemma II.1.** If $\Re \zeta \geq 0$, then problem (1) - (4) has no more than one solution, and its solution does exist.

Let us show that the solution to problem (1) - (4) can be found in the form $E = E_0 + e$, where $e$ is given in formula (5).

**Lemma II.2.** The solution to problem (1) - (4) is representable in the form $E = E_0 + e$, where $e$ is given by formula (5) with a uniquely determined tangential to $S$ field $J$.

**Proof:** By Lemma II.1 there exists a unique solution to problem (1) - (4). Let $[N,E]|_S$ be known. Then the solution $E$ to problem (1) - (4) is uniquely determined in $D'$. Therefore, Lemma II.2 will be proved if one checks that the solution to problem (1) - (4) has boundary values $[N,E]|_S$ which can be represented by the boundary values of the function

$$E_0(x) + \nabla \times \int_S g(x,t) J(t) dt|_{x \rightarrow s^-},$$

where $x \rightarrow s^-$ means that $x \rightarrow s$ from $D'$.

Note that the function $E_0(x) + \nabla \times \int_S g(x,t) J(t) dt$ satisfies equations (1) - (3) for any $J(t)$.

Let us check that there is a unique $J$, tangential to $S$, such that

$$E_0(s) + e(s) = E_0(s) + \nabla \times \int_S g(x,t) J(t) dt|_{x \rightarrow s^-},$$

$$\lim_{x \rightarrow s^-} [N, \nabla \times \int_S g(x,t) J(t) dt] = [N_s, e(s)]]$$

(9)

where $e = e(s) := E(s) - E_0(s)$, and $[N_s, e(s)]$ is assumed to be known. Let us use the known formula (see, for example, [5], p. 158):

$$\lim_{x \rightarrow s^-} [N_s, \nabla \times \int_S g(x,t) J(t) dt] =$$

$$\frac{J(s)}{2} + \int_S [N_s, \nabla g(s,t), J(t)] dt.$$  

(10)

From (9) - (10) it follows that

$$\frac{J(s)}{2} + TJ := \frac{J(s)}{2} +$$

$$\int_S [N_s, \nabla g(s,t), J(t)] dt = [N_s, e(s)].$$

(11)

This is an integral equation for $J$, and the right-hand side $[N_s, e(s)]$ is known. Let us prove that (11) is uniquely solvable for $J$ if $D$ is sufficiently small. It is sufficient to check that the norm in $C(S)$ of the integral operator $T$ in (11) tends to zero as $a \rightarrow 0$.

One has

$$||TJ|| \leq (\max_{s \in S} \int_S |\nabla_s g(s,t)|) \left|N_s \cdot \frac{J(t)}{||J(t)||}\right| dt +$$

$$\max_{s \in S} \int_S \left|\frac{\partial g(s,t)}{\partial N_s}\right| dt ||J||.$$  

(12)

Since $S$ is a smooth surface and $J$ is tangential to $S$, one has

$$\left|N_s \cdot \frac{J(t)}{||J(t)||}\right| \leq c |s - t|, \quad \left|\frac{\partial g(s,t)}{\partial N_s}\right| \leq c |s - t|^{-1},$$

(13)

where $c > 0$ stands for different constants depending only on $S$.

From estimates (12) and (13) it follows that

$$||T|| \leq O(a), \quad a \rightarrow 0.$$  

(14)

Thus, $||T|| < 1$ for sufficiently small $a$, and equation (11) determines $J$ uniquely. Clearly, $N_s \cdot J(s) = 0$, that is, $J$ is tangential to $S$.

The knowledge of the tangential component of the scattered field $e(x)$ on $S$ determines this field $e(x)$ everywhere in $D'$ if this field satisfies equations (1) - (3). Therefore the field $e(x)$ can be represented in the form (5) with $J$ tangential to $S$. Lemma 2.2 is proved.

Assume that the origin is inside $S$. If $ka \ll 1$ then the scattered field $e(x)$ at the distances $|x| \gg a$ can be calculated by the first formula (6) with $x_1 = 0$.

**Lemma II.3.** Second formula (6) holds.

**Proof:** A proof one can find in [5], pp. 86 - 91. We want to present in more detail the estimate

$$\left|\frac{\partial^2 J_m(t)}{\partial t_m \partial t_p}\right| \leq c,$$  

(15)

used in the proof. Here $J_m$ is the $m$-th component of $J$.

From equations (1) it follows that

$$Le := (\nabla^2 + k^2)e = 0 \text{ in } D',$$  

(16)

$$\nabla \cdot e = 0 \text{ in } D',$$

$e$ satisfies the radiation condition (3), and the impedance boundary condition (4). The known term $f$ in (4) is smooth: if
If $E_0 \in H^i(S)$ then $f \in H^{i-1}(S)$, where $H^i(S)$ is the standard Sobolev space.

If $h := \frac{\nabla \times e}{\sqrt{\varepsilon \mu}}$ and $k = \omega \sqrt{\varepsilon \mu}$, then (16) implies equations (1) for $e$ and $h$. Indeed, $\nabla \times \nabla \times e = -\nabla^2 e$ in $D'$ because $\nabla \cdot e = 0$ in $D'$. Thus, $\nabla \times \nabla \times e = i\omega \mu \nabla \times h = k^2 e$. Consequently,

$$\nabla \times h = -i\omega e, \quad \nabla \times e = i\omega \mu h,$$

as claimed.

Equation (16) is elliptic. In the proof of Lemma II.2 it was established that $J$ is as smooth as $[N_s, e(s)]$. From the elliptic regularity for the solutions to equation (16), it follows that the solution $e$ is two derivatives smoother than the $f$ in (4). Since $E_0$ is infinitely smooth and the surface $S$ is assumed smooth, one concludes that $f$ and $[N_s, e(s)]_S$ are smooth. Therefore, $J$ is smooth, and estimate (16) is justified.

From Lemmas II.1 - II.3 the result of this paper follows:

**Theorem II.4.** If $ka \ll 1$ and $S$ is smooth then formulas (6) hold.