A DESIGN ALGORITHM USING Z-PLANE CLOSED LOOP POLE PLACEMENT

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ABSTRACT

Design of a Single Input Single output (SISO) control system is simplified if the designer has direct control over the closed loop poles. This paper presents a design example utilizing a Z-transform algorithm that constructs a generic discrete Compensator. The "Plant" in the example has an anti-aliasing filter, a double integrator, a torquers time constant, and a sample and hold as continuous elements: and a discrete proportional plus integral module. This 5th order system is representative of many instrument pointing control systems. Poles resulting from the Compensator are placed near the origin of the Z-plane so that the response is dominated by the closed loop poles from the "Plant". Root locus, time response, and frequency response data are given. Normalization by the sample interval, T_s, is used to provide a dimension-less example.

INTRODUCTION

Designers of SISO control systems during the 1960's were taught to model systems with Laplace transforms using texts such as Gardner and Barnes(1). Once the open loop transfer function of the "Plant" was derived, techniques from texts such as Truxal(2) facilitated the design of compensators to stabilize and enhance the performance of the system. Beginning with the known pole-zero locations of the "Plant", these techniques directed the designer how to choose the open loop pole-zero locations of the compensator. Open loop Bode frequency plots and the graphical root locus are two such techniques used to approximately position the closed loop poles. Only with simple 2nd order systems could the closed loop poles be specified directly.

The techniques developed for continuous systems were applied to discrete systems in such texts as Tou(3). The mathematical power for this was the Z-transform which mapped the jw axis of the S-plane into the unit circle of the Z-plane. Keeping the closed loop poles inside the unit circle replaced the goal of keeping all the poles in the left half plane.

In the 1970's and 80's state variable models of systems began to replace or augment traditional Laplace transform models. Design techniques using weighted quadratic performance indexes of the model states were introduced as a direct means of insuring stability and specifying performance. This technique was directed towards removing some of the "trial and error" used in the design cycle; however, it also lost some of the insight into how the system worked and how it would function if the model was not perfect.

In texts such as Oppenheim and Willsky(4) continuous and discrete signals are presented in an integrated fashion, including frequency response transfer functions for both types of signals.

In Franklin and Powell(5) the state variable technique for direct closed loop pole-zero placement is presented. It is a multistep process involving forming a state feedback control law and then a state estimator. These two can be combined into a compensator design.

The technique in this paper is a method of arriving at the same compensator design using only the Z-plane model of the Plant and a generic Compensator. For SISO systems it appears to be a more direct algorithm. Also since Z-transforms are generated, it is a simple additional step to produce closed loop frequency response data and root locus data in order to examine these characteristics of system performance.

CLOSED LOOP POLE PLACEMENT

THE GENERIC COMPENSATOR

The Compensator is defined as a lead weighting vector applied to the last n output samples plus a "lag" weighting vector applied to the last m control levels where "n" is the order of the Plant, and m = n-1.

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The control \( u_c[k] \) at time \( k*T \) is
\[
- u_c[k] = X_{lk}B_0 + X_{lk-1}B_1 + \ldots + X_{lk-m}B_m + u_c[k-1]A_1 + \ldots + u_c[k-m]A_m
\]
where \( B_i \) for \( i = 0, 1, \ldots, m \) are the control elements applied to the string of samples of the output state, \( X_1 \); and \( A_i \) for \( i = 1, 2, \ldots, m \) are control elements applied to the string of previous control levels, \( U_{k-1}, \ldots, U_{k-m} \).

Taking the Z-transform of equation (1)
\[
- U_c(Z) = X_l(Z)B_0 + B_1Z^{-1} + \ldots + B_mZ^{-m} + U_c[Z]A_1 + \ldots + A_mZ^{-m}
\]
and forming the ratio \( U_c(Z)/X_l(Z) \) yields the transfer function
\[
G_c(z) = \frac{BO*Z^m + B_1*Z^{m-1} + \ldots + B_m}{Z^m + A_1*Z^{m-1} + \ldots + A_m}
\]
(3)

**NOTE:** In this paper **"*** is multiplication.

**THE CLOSED LOOP POLYNOMIAL**

The Z-transform of the Plant can be, in general, expressed as
\[
G_p(z) = g* \frac{b_0*z^m + b_1*z^{m-1} + \ldots + b_m}{a_0 + a_1*z^{-1} + \ldots + a_n}
\]
(4)

where "g" is the open loop gain. Even if the S-plane Plant model has no zeros, there will be zeros in the transfer function of the Z-plane model. The number of zeros will be one less than the number of poles. The number of poles is the same in both cases.

Combining the Z-plane models of the Compensator and of the Plant, the closed loop polynomal
\[
1 + G_c(z) = 0
\]
(5)
can be formed. Using \( G_c(z) \) given in Equation (3), the polynomial expands to
\[
2^m( z^n + a_1*z^{n-1} + \ldots + a_n )
+ A_1*Z^{-1}( z^n + a_1*z^{n-1} + \ldots + a_n )
+ A_2*Z^{-2}( z^n + a_1*z^{n-1} + \ldots + a_n )
\ldots
Am*( z^n + a_1*z^{n-1} + \ldots + a_n )
\]
plus
\[
g*B_0*Z^m( b_0*z^m + b_1*z^{m-1} + \ldots + b_m )
g*B_1*Z^{m-1}( b_0*z^m + b_1*z^{m-1} + \ldots + b_m )
g*B_2*Z^{m-2}( b_0*z^m + b_1*z^{m-1} + \ldots + b_m )
\ldots
\]
\[
g*B_m( b_0*z^m + b_1*z^{m-1} + \ldots + b_m )
\]
equals 0.

Forming the closed loop polynomial as the product of the desired closed loop poles yields
\[
(Z - P_1)(Z - P_2) \ldots *(Z - P_q) = z^q + d_1*z^{q-1} + \ldots + d_q = 0
\]
(7)
with \( q = (n + m) \).

**THE DESIGN MATRIX**

For Equations (4) and (5) to be identical, the coefficients of the powers of \( Z \) must be equal. Equating these produces a linear set of equations represented by the following matrix equation.
\[
\begin{bmatrix}
1 & g*B_0 & g*B_1 & \ldots & g*B_m & a_1 & a_2 & \ldots & a_n \\
0 & 0 & 0 & \ldots & 0 & b_0 & b_1 & \ldots & b_m \\
1 & 0 & 0 & \ldots & 0 & a_1 & a_2 & \ldots & a_n \\
0 & 0 & 0 & \ldots & 0 & b_0 & b_1 & \ldots & b_m \\
\end{bmatrix}
\]
(8)

with
\[
\begin{bmatrix}
b_0 & b_1 & b_2 & \ldots & b_m & a_1 & a_2 & \ldots & a_n \\
0 & 0 & 0 & \ldots & 0 & b_0 & b_1 & \ldots & b_m \\
1 & 0 & 0 & \ldots & 0 & a_1 & a_2 & \ldots & a_n \\
0 & 0 & 0 & \ldots & 0 & b_0 & b_1 & \ldots & b_m \\
\end{bmatrix}
\]

The first row of the matrix equates the coefficients of \( z^{q-1} \) and the last row those of \( z^0 \).

The Design Matrix is set up in the following manner. The columns are the coefficients \( b_0, b_1, b_2, \ldots, b_m \) corresponding to the polynomial determining the open loop Plant zeros and the coefficients \( a_1, a_2, \ldots, a_n \) corresponding to the polynomial determining the open loop Plant poles. Fill zeros are added as shown in Equation (9) to complete the columns.

**THE DESIGN MATRIX INVERSE**

The matrix has an inverse as long as Control-ability and Observe-ability are present. For example if there is zero pole cancellation in \( G_p(z) \), a state cannot be observed resulting in a singular matrix.

The Compensator elements are given by
\[
\begin{bmatrix}
g*B_0 & g*B_1 & \ldots & g*B_m \\
\ldots & \ldots & \ldots & \ldots \\
g*B_m \\
\end{bmatrix}
\]

with \( g*B_m = DSGM* \)
\[
\begin{bmatrix}
d_1 & a_1 & a_2 & \ldots & a_n \\
0 & 0 & 0 & \ldots & 0 \\
1 & 0 & 0 & \ldots & 0 \\
0 & 0 & 0 & \ldots & 0 \\
\end{bmatrix}
\]
(10)

Note that the "lead" control elements, \( B_0, B_1, \ldots, B_m \) are multiplied by the open
The Z-transform of $G_p(s)$, which includes the sample and hold, is derived as follows.

$$G_p(s) = \frac{(1 - \exp^{-TS})}{S}$$

$$G_p(z) = \frac{Z - 1}{Z} \cdot \text{transform}(G_p(s)/S).$$

For this example,

$$G_p(s) = \frac{a*b}{K \cdot S^2 \cdot (S + a)(S + b)}$$

where "$K$" is the physical Plant gain and both the filter and the torquer models are set to have unity gain.

**PARTIAL FRACTION EXPANSION**

The partial fraction coefficients of

$$G_p(s) = \frac{k_{11} \cdot k_{12} \cdot k_{13} \cdot k_{21} \cdot k_{31}}{K \cdot S^2 \cdot (S + a)(S + b)}$$

using the method described in Gardner and Barnes (1), are

$$k_{11} = 1$$

$$k_{12} = -((1/aT) + (1/bT))$$

$$k_{13} = \frac{1}{T^2} \cdot (aT)^2$$

$$k_{21} = \frac{1}{T^2} \cdot (a/bT)$$

$$k_{31} = \frac{(a/bT)^2}{T^2}$$

and using the elementary Z-transform pairs found in Tou (3)

$$G_p(z) = \frac{k_{11} \cdot k_{12} \cdot k_{13} \cdot k_{21} \cdot k_{31}}{K \cdot T^2 \cdot (Z - 1)^3}$$

$$+ \frac{Z}{T \cdot (Z - 1) \cdot (Z - \exp^{-aT})}$$

**NOTE:** $aT = a \cdot T$, $bT = b \cdot T$, and $K T^2 = K \cdot T^2$
COMPLETING THE MODEL

Using equation (15),

\[
\frac{Gp(z)}{kT^2} = \frac{1}{z - 1} + \frac{\text{---}^* (k_{13} + k_{21})}{T^2} \left( z - \text{exp}^{-\alpha T} \right)
\]

\[
= \frac{\text{---}^* (k_{11} - \text{exp}^{-\beta T})}{z - \text{exp}^{-\beta T}}
\]

\[\text{(21)}\]

NOTE: (k_{13} + k_{21} + k_{31} = 0) when combining the last three terms of Equation (21).

No further algebraic reduction is necessary. The coefficients of the two polynomials whose ratio correspond to \(Gp(z)\) are quickly calculated using the "POLY" features found in many Personal Computer (PC) math packages. The final step before applying the Design Matrix is to include the integrator module to complete the open loop Plant model.

\[
\frac{Z - Z_0}{Gp(z)} = g*\frac{1}{z - 1} \cdot Gp(z)
\]

\[\text{(22)}\]

where the normalized, nominal open loop gain is

\[
g = \frac{(1 + g_i)kT^2}{1} = 1
\]

\[\text{(23)}\]

and

\[
Z_0 = \frac{1}{(1 + g_i)}. \quad \text{(24)}
\]

NOTE: In the root locus data given, the value of \(g_i = 1\) corresponds to the poles set by the Design Matrix.

In the design example the time constant of the anti-aliasing filter is "\(bT\)" and the value of the torquer time constant is "\(\alpha T\)". The value of the zero, \(Z_0\), created in the Plant gives the designer an extra degree of freedom as shown later in the example.

DESIGN CHOICES

Let the Plant time constants to be given as, \(bT = 1.0\) and \(\alpha T = 0.1\). Then using Equation (22), the open loop Plant zeros and poles are located at

\[
\begin{array}{c|c|c}
Z_1 & -8.1251 & P_1 \\
Z_2 & -0.8049 & P_2 \\
Z_3 & -0.0791 & P_3 = 1.0000 \\
Z_4 & Z_0 & P_4 = 0.9048 \\
\end{array}
\]

\[
\begin{array}{c|c|c}
P_5 & 0.3679 \\
\end{array}
\]

NOTE: Arbitrarily setting the integrator gain at \(g_i = 4\) creates a Plant zero at \(Z_0 = 0.2\).

There are five closed loop poles to choose. Select two complex pairs, one with a S-plane natural frequency, \(\omega T\), at \((\pi/10)\) and the second at \((\pi/3)\). Select the 5th pole to have a value equal to the absolute value of the \((\pi/10)\) pair. Let both complex pairs have a S-plane damping factor of 0.7. Beginning with the S-plane location of a complex pole,

\[
S_1T = -0.7*\omega T + j[(1 - .72)*.5]*\omega T
\]

and moving it into the z-plane using

\[
z = \exp^{T*S}
\]

\[\text{(25)}\]

yields

\[
Z_1 = (\exp^{-0.7*pi/10})*(\exp{j*0.7141*pi/10})
\]

\[
Z_1 = 0.8026*(\cos(12.85^0) + j*\sin(12.85^0)).
\]

The z-plane location of the desired closed loop poles of the Plant are then

\[
\begin{array}{c|c|c}
0.7825 + j*0.1786 \\
0.7825 - j*0.1786 \\
0.8026 \\
0.3522 + j*0.3267 \\
0.3522 - j*0.3267
\end{array}
\]

There are four closed loop Compensator poles to choose. It terms of the state variable approach to pole placement, these poles determine how rapidly the state estimator converges.

Select a complex pair with \(\omega T = (\pi/1.5)\) and a S-plane damping factor of 0.7. Select the third pole equal to the absolute value of the complex pair. Place the fourth pole at \(Z = 0\). The resulting desired \(z\)-plane locations for the Compensator poles are

\[
\begin{array}{c|c|c}
0.0173 + j*0.2302 \\
0.2308 \\
0.0000
\end{array}
\]

The Design Matrix can now be used to create a Compensator with elements

\[
\begin{array}{c|c|c}
B_0 & 32.669 \\
B_1 & -95.852 \\
B_2 & 103.137 \\
B_3 & -47.463 \\
B_4 & 7.643 \\
A_1 & 0.8247 \\
A_2 & 0.3986 \\
A_3 & 0.083 \\
A_4 & 0.0000
\end{array}
\]

or

\[
B_0 = 32.669 + j*0.0000 \\
B_1 = -95.852 - j*0.0000 \\
B_2 = 103.137 + j*0.0000 \\
B_3 = -47.463 - j*0.0000 \\
B_4 = 7.643 + j*0.0000
\]

or

\[
A_1 = 0.8247 + j*0.0000 \\
A_2 = 0.3986 - j*0.0000 \\
A_3 = 0.083 - j*0.0000 \\
A_4 = 0.0000 + j*0.0000
\]

for \(Z_0 = 0.2000\) or \(Z_0 = 0.840552\).

The 1st set are for the arbitrarily selected value of \(g_i\) and the 2nd set is with \(g_i\) and the resulting Plant zero tuned to result in \(B_4 = A_4 = 0\).
This reduces the dimension of the Compensator to 3rd order with zeros and poles

\[
\begin{align*}
Z_1 &= 0.8717 + j*0.1455 \\
Z_2 &= 0.8717 - j*0.1455 \\
Z_3 &= 0.3638 \\
P_1 &= -0.3886 + j*0.5819 \\
P_2 &= -0.3886 - j*0.5819 \\
P_3 &= -0.0630
\end{align*}
\]

and also eliminates the closed loop pole at z = 0.

**PERFORMANCE CHARACTERISTICS**

**ROOT LOCUS DATA**

The root locus data for this Compensator is plotted in Figure 1 for a range of \( g \) = [0.6, 0.7, 0.8, 0.9, 0.95, 1.0, 1.05, 1.1, 1.2, 1.3, 1.4]

![Figure 1 - Root Locus Plot](image)

**TIME RESPONSE**

The state transition matrix, using a T/2 time increment, for the transfer function of Equation (17) plus a sample and hold is

\[
X_{k+1} = \exp\left(\frac{1}{2}AT\right)X_k
\]

with

\[
AT = \begin{bmatrix}
-bT & bT & 0 & 0 & 0 & 0 \\
0 & 0 & 1 & 0 & 0 & 0 \\
0 & 0 & 0 & 1 & 0 & 1 \\
0 & 0 & 0 & -aT & aT & 0 \\
0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0
\end{bmatrix}
\]

The states in the model are (1) the output of the filter, (2) the mirror angle, (3) the angular rate, (4) the torquer output, (5) the sample and hold output, and (6) an acceleration step disturbance. The last 2 rows are all zeros since these states do not change value during the T/2 interval. All rates are normalized by T and all accelerations by T^2.

Equation (26) was used to simulate the design example and produce the time response data shown in Figure 2. The "+" time responses correspond to the loop gains limits of 0.6 and 1.4 used in the root locus plot. The solid plot is the response with \( g = 1 \). The response is to an acceleration disturbance of 10 units.

![Figure 2a - Output Time Response g = 0.6](image)

![Figure 2b - Output Time Response g = 1.4](image)

The magnitude of the two complex pole pairs that dominate the time response are given below. Selecting 1.2 > g > 0.8 results in a nearly constant time response.

**TABLE I - DOMINANT CLOSED LOOP POLES**

<table>
<thead>
<tr>
<th>Loop Gain</th>
<th>Low Freq Magnitude</th>
<th>High Freq Magnitude</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.6</td>
<td>0.5550</td>
<td>0.9511</td>
</tr>
<tr>
<td>0.8</td>
<td>0.8646</td>
<td>0.4717</td>
</tr>
<tr>
<td>1.0</td>
<td>0.8026</td>
<td>0.4799</td>
</tr>
<tr>
<td>1.2</td>
<td>0.8374</td>
<td>0.7850</td>
</tr>
<tr>
<td>1.4</td>
<td>0.8502</td>
<td>0.9387</td>
</tr>
</tbody>
</table>
FREQUENCY RESPONSE
When a base motion disturbance moves the laser beam off its null, the control follows this motion and works to maintain zero error. Using the methods described in Tou[3], the Z-plane transfer function for this disturbance, D(s), is

\[
G_c(z)D(z)G_p(z) = \frac{1 - D G_p^2(z)}{1 + G_p(z)}
\]

Before being sampled, a sinusoidal disturbance is attenuated and shifted by the anti-aliasing filter by the amount

\[
\frac{bT}{|\omega T|/\pi}
\]

DG_p(z) is then this signal sampled. The disturbance enters before the filter, so

\[
G_p(s) = \frac{1}{s^{2}(s + a)}
\]

Letting \( z = e^{j\omega T} \) yields the discrete Fourier transform of Equation (27) shown below. For \( |\omega T|/\pi < 1/2 \), it is a good approximation of the frequency response characteristic of the continuous output of the Plant to a sinusoidal disturbance.

**Table II - Discrete Fourier Transform**

<table>
<thead>
<tr>
<th>FREQUENCY</th>
<th>PHASE</th>
<th>MAGNITUDE (ratio)</th>
<th>MAGNITUDE (Log10)</th>
</tr>
</thead>
<tbody>
<tr>
<td>1/2</td>
<td>37.0</td>
<td>1.83</td>
<td>0.2623</td>
</tr>
<tr>
<td>1/2.5</td>
<td>11.4</td>
<td>2.75</td>
<td>0.4401</td>
</tr>
<tr>
<td>1/3</td>
<td>13.4</td>
<td>3.19</td>
<td>0.5043</td>
</tr>
<tr>
<td>1/4</td>
<td>53.1</td>
<td>3.15</td>
<td>0.4977</td>
</tr>
<tr>
<td>1/5</td>
<td>82.4</td>
<td>2.69</td>
<td>0.4295</td>
</tr>
<tr>
<td>1/6</td>
<td>105.4</td>
<td>2.22</td>
<td>0.3457</td>
</tr>
<tr>
<td>1/8</td>
<td>141.5</td>
<td>1.46</td>
<td>0.1650</td>
</tr>
<tr>
<td>1/10</td>
<td>169.4</td>
<td>0.94</td>
<td>0.0256</td>
</tr>
<tr>
<td>1/25</td>
<td>-112.7</td>
<td>0.0617</td>
<td>-1.2098</td>
</tr>
<tr>
<td>1/100</td>
<td>-91.4</td>
<td>0.00072</td>
<td>-3.1406</td>
</tr>
<tr>
<td>1/1000</td>
<td>-90.1</td>
<td>0.0000000</td>
<td>-6.1566</td>
</tr>
</tbody>
</table>

Note that attenuation of the disturbance occurs for \( |\omega T|/\pi < 1/10 \) and approaches the limit of 3 decades per decade as the frequency approaches zero. This is the expected result with 3 integrators in the open loop transfer function.

For \( |\omega T|/\pi > 1/2 \) the simulation model shows that the response tends toward unity magnitude and zero phase.

Figure 3 shows the response of the simulation model for an input sinusoidal disturbance at \( \omega T = \pi/25 \). The solid line is the disturbance and the "+" line is the output response.

**CONCLUSIONS**
Equation (10) gives the system designer a Z-transform based design tool that will place the closed loop poles anywhere in the Z-plane. Working with the Z-transform of a SISO system, the compensator is quickly constructed from the open loop poles and zeros of the Plant. Time, frequency, and root locus data can then be generated to characterize the system performance for the selected closed loop poles.

This method, easily implemented on present day PC's, enables the system designer to pick the location of the system closed loop poles based on "judgement" factors. This retains and develops much of the insight of classical techniques into how feedback systems operate and what affects system performance.

**REFERENCES**
(1) Gardner and Barnes (1942), Transients in Linear Systems, John Wiley & Sons, New York, NY