Approximate solution of the Fitzhugh-Nagumo equation by the homotopy analysis method

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Abstract—In this paper, we used a different initial guess solution then others, to obtain very good approximate solution of the Fitzhugh-Nagumo equation by the homotopy analysis method. Thus, we further illustrate the freedom to select the initial guess solution, i.e. illustrate effectiveness and convenience of homotopy analysis method.

Keywords-component: Homotopy analysis method, Fitzhugh-Nagumo equation, Approximate solution

I. INTRODUCTION

Nonlinear equations are widely used as models to describe complex physical phenomena in various fields of sciences. The world around us is inherently nonlinear. Nonlinear problems are more difficult to solve than linear ones. Recently, a powerful analytic method for nonlinear problems, the so-called homotopy analysis method (HAM), has been proposed by Liao [1]. Moreover, the homotopy analysis method itself provides us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary. Briefly speaking, the homotopy analysis method has the following advantages:

1. it is valid even if a given nonlinear problem does not contain any small/large parameters at all;
2. it itself can provide us with a convenient way to control the convergence of approximation series and adjust convergence regions when necessary;
3. it can be employed to efficiently approximate a nonlinear problem by choosing different sets of base functions.

Furthermore, the HAM has been successfully applied to many nonlinear problems such as non-linear eigenvalue problem[2], viscous flows [3], and heat transfer [4], nonlinear oscillations [5], Thomas-Fermi’s atom model [6]-[7], the Kawahara equation[8], nano boundary layer flows[9], solitary-wave solutions for the fifth-order KdV equation[10], and many other problems(see[11]-[17], for example). And some elegant analytic results are obtained. All of these successful applications of the homotopy analysis method verify its validity for nonlinear problems in science and engineering.

II. MATHEMATICAL FORMULATION

The Fitzhugh-Nagumo equation is

\[ u_t - u_{xx} = u(u - \alpha)(1 - u), \]  

where \( \alpha \) is an arbitrary constant. Eq. (1) is an important nonlinear reaction-diffusion equation and applied to model the transmission of nerve impulses[18], [19], also used in biology and the area of population genetics, in circuit theory[20].

Under the transformation \( \xi = x - ct \) and \( u(x, t) = U(\xi) \), Eq. (1) reads

\[ U'' + cU' + U(U - \alpha)(1 - U) = 0, \]  

where \( c \) is the velocity of the travelling wave and the prime denotes the derivative with respect to \( \xi \).

We should consider the solitary waves. Obviously, \( U(\xi) \) and its derivatives tend to zero as \( \xi \to \infty \). Hence, the boundary conditions of the solitary wave are

\[ U(0) = 1, U(+\infty) = 0. \]  

III. SOLUTION BY THE HAM

S. Abbasbandy[21] solved this problem with same technique, he choose \( U_0 = \exp(-\xi) \) as the initial guess solution. In this paper, we use a different initial guess solution, the concrete procedure is as follows.

According to the Eq. (2) and the boundary conditions (3), the solitary solution can be expressed by

\[ U(\xi) = \sum_{k=1}^{+\infty} b_k \exp(-k\xi). \]  

According to the solution expression (4) and the boundary conditions (3), the initial guess solution of \( U(\xi) \) is chosen as the following

\[ U_0 = 2 \exp(-\xi) - \exp(-2\xi). \]  

Thus, we select the auxiliary linear operator

\[ L[\Phi(\xi, q)] = \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial \Phi}{\partial \xi}. \]  

From (2), we define the nonlinear operator

\[ N[\Phi] = \frac{\partial^2 \Phi}{\partial \xi^2} + \frac{\partial \Phi}{\partial \xi} - \alpha \Phi + (1 + \alpha) \Phi^2 - \Phi^3. \]

Constructing the so-called zeroth-order deformation equation

\[ (1 - q)L[\Phi(\xi, q) - U_0(\xi)] = qhH(\xi)N[\Phi(\xi, q)], \]  

where \( q \) is an arbitrary parameter.
subject to the boundary conditions

$$\Phi(0, q) = 1, \Phi(\infty, q) = 0,$$

where \(q \in [0, 1]\) is the embedding parameter, \(h \neq 0\) is an auxiliary parameter, \(H(\xi)\) is an auxiliary function, \(\Phi(\xi, q)\) is an unknown function. It is obvious that when the embedding parameter \(q = 0\) and \(q = 1\), the equation (7) become

$$\Phi(\xi, 0) = U_0(\xi), \Phi(\xi, 1) = U(\xi),$$

$$C(0) = c_0, C(1) = c,$$

respectively. Therefore, when \(q\) increases from 0 to 1, the solution \(\Phi(\xi, q)\) varies from the initial guess solution \(U_0(\xi)\) to the exact solution \(U(\xi)\), so does the \(C(q)\) from \(c_0\), the initial guess of the wave speed, to \(c\). Expanding \(\Phi(\xi, q)\) and \(C(q)\) in Taylor series with respect to \(q\), we have

$$\Phi(\xi, q) = U_0(\xi) + \sum_{k=1}^{\infty} U_k(\xi)q^k,$$

$$C(q) = c_0 + \sum_{k=1}^{\infty} c_k q^k,$$

where

$$U_k(\xi) = \frac{1}{k!} \frac{\partial^k \Phi(x, q)}{\partial q^k} \bigg|_{q=0},$$

$$c_k = \frac{1}{k!} \frac{\partial^k C(q)}{\partial q^k} \bigg|_{q=0}.\,$$

The convergence of the series (11)-(12) depends upon the auxiliary parameter \(h\). If it is convergent at \(q = 1\), we have

$$U(\xi) = U_0(\xi) + \sum_{k=1}^{\infty} U_k(\xi)$$

$$c = c_0 + \sum_{k=1}^{\infty} c_k$$

which must be one of the solutions of the original nonlinear equation. Define the vectors

$$U_k = \{U_0(\xi), U_1(\xi), \cdots, U_k(\xi)\}$$

Differentiating the zeroth-order deformation Eq.(7) \(k\)-times with respect to \(q\) and then dividing them by \(m!\) and finally setting \(q = 0\), we obtain the \(k\)th-order deformation equation

$$L[U_k(\xi) - \chi_k U_{k-1}(\xi)] = hH(\xi)R_k(U_{k-1}),$$

subject to

$$U_k(0) = 0, U_k(\infty) = 0,$$

where

$$R_k(U_{k-1}) = U'' - \alpha U' + \sum_{i=0}^{k-1} (c_{k-1-i} U'_i)$$

$$+ \sum_{i=0}^{k-1} ((1 + \alpha)U_i U_{k-1-i} - U_{k-1-i} \sum_{j=0}^{i} U_j U_{i-j}),$$

and

$$\chi_k = \begin{cases} 0, & m \leq 1, \\ 1, & m > 1. \end{cases}$$

In \(R_k(U_{k-1})\), the coefficient of \(\exp(-\xi)\) is must vanished, otherwise the solution \(U_k\) of Eq. (17)-(18) should contain the so-called secular term \(\exp(-\xi)\). Thus, determining \(c_{k-1}\). So, any unknown parameters are not contained in \(R_k(U_{k-1})\).

Let \(U_k^*(\xi)\) denote a special solution of equation

$$L[U_k^*(\xi)] = hH(\xi)R_k(U_{k-1}).$$

Because, the general solution of \(L[\phi] = 0\) is

$$\phi = C_1 \exp(-\xi) + C_2.$$

Thus, the general solution of Eq. (17) can be expressed as

$$U_k(\xi) = \chi_k U_{k-1}(\xi) + U_k^*(\xi) - U_k^*(0) \exp(-\xi),$$

In this way, we derive \(U_k\) for \(m = 1, 2, \cdots, \) successively. At the \(M\)th-order approximation, we have the analytic solution of Eq. (2)-(3), namely

$$U(\xi) \approx W_M = \sum_{k=0}^{M} U_k(t).$$

IV. CONCLUSION

Form (2), we define the error as follow

$$Error \approx W_M(W_M - \alpha)(1 - W_M) + W_M' + cW_M'.$$

When \(\alpha = 1.5\), from the \(h\)-curves, Fig.1, we find that \(h \in [-1.4, -0.5]\). If we select \(h = -1\), obtain the error of approximate solution as show Fig.2.

When \(\alpha = 0\), from the \(h\)-curves, Fig.3, we find that \(h \in [-1.1, -0.7]\). If we select \(h = -1\), obtain the error of approximate solution as show Fig.4.

![Fig.1. h-curves for \(\alpha = 1.5\), solid line: the error of 15th-order approximation at \(\xi = 1\); dashed line: the error of 15th-order approximation at \(\xi = 2\).](image-url)
Fig.2. Error for $\alpha = 1.5$, solid line: 15th-order analytical approximate solution; dashed line: 10th-order analytical approximate solution.

Fig.3. $h$-curves for $a = 0$, solid line: the error of 15th-order approximation at $\xi = 1$; dashed line: the error of 15th-order approximation at $\xi = 2$.

Fig.4. Error for $\alpha = 0$, solid line: 10th-order analytical approximate solution; dashed line: 15th-order analytical approximate solution.

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