A Coupled Oscillators-based Control Architecture for Locomotory Gaits

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Abstract—This paper presents a bio-inspired central pattern generator (CPG) architecture for optimal control of locomotory gaits. The CPG circuit is realized as a coupled oscillator feedback particle filter. The collective dynamics of the filter are used to approximate a posterior distribution that is used to construct the optimal control input.

The architecture is illustrated with the aid of a model problem involving locomotion of coupled planar rigid body systems, with two links. For this problem, the coupled oscillator feedback particle filter is designed and its control performance demonstrated in a simulation environment.

I. INTRODUCTION

This paper is concerned with the problem of optimal control of locomotory gaits, based on noisy sensor measurements. The motivation comes from the problem of sensorimotor control of locomotion, cf., [13]. In many biologically inspired robotic systems, locomotion is effected by periodic actuation of the robot’s internal degrees of freedom, which consequently induces a global displacement of the system. This approach has been applied to systems as varied as winged robots performing flapping flight [27], fish-like swimming robots [31], [6], micro-swimmers [4], [32], snake-like robots [11], [23], [7], and legged robots [9], [10], [13]. Even wheeled locomotion can be described in this way by explicitly defining the rotation of the wheels as the internal degrees of freedom.

In systems such as these, it is often the case that the configuration variables of the system can be partitioned into internal variables whose dynamics are governed by a set of second order differential equations, and group variables, whose evolution is governed by a set of first-order equations. The internal variables define the shape of the robot, and thus, are typically referred to as the shape variables; the group variables uniquely define the pose of the robot body frame, and thus typically belong to SE(n) or one of its subgroups.

In this paper, we consider a special class of such systems, in which the shape variables are driven by an open-loop periodic input, \( \tau(t) \), and the evolution of the group variables depends upon the shape variables as well as an externally applied control input, \( u(t) \). We assume only the availability of noisy observations of the shape variables, and derive control laws that optimize various locomotion criteria (e.g., absolute change in orientation, or translational motion along a particular degree of freedom). We provide a general solution methodology for such problems, which comprises modeling, estimation and control, and we illustrate the approach on one specific problem of turning (rotating) a two-body planar system.

Here we provide a high-level overview of the solution for the problem, which we present in more detail in Sec. II and Sec. III. The overall system architecture is shown in Fig. 1. The robot is the two-body system depicted in Fig. 2 with a primary body (head) and a secondary body (tail). The shape variable \( x \) denotes the relative orientation of the tail with respect to the head, and the group variable \( q_1 \) denotes the absolute orientation of the head. We consider the control objective of maximizing the change in \( q_1 \) (the orientation of the head) by actuating the length of the tail body (i.e., \( u(t) \) effects a change in \( d_2 \)). A periodic open-loop torque \( \tau \) is applied by a single motor at the joint connecting the two links, and this results in both \( x \) and \( q_1 \) oscillating in a periodic (but uncontrolled) manner.

The general solution methodology proposed in this paper comprises of three parts:

1. Modeling: The modeling objective is to obtain a reduced order phase model of the dynamics. The instantaneous phase variable, denoted as \( \theta(t) \), is a stochastic process taking values on the circle \([0,2\pi]\). For the two-body model problem, the phase variable is the coordinate of the limit cycle solution for \((x, \dot{x})\) (see Fig. 4); cf., [15], [17], [3].

2. Estimation using CPG: The estimation problem is to construct a nonlinear filter for estimating the instantaneous phase \( \theta(t) \), given a time-history of noisy measurements. The problem is solved numerically by constructing a coupled oscillator feedback particle filter; cf., [24], [26]. The coupled oscillator filter comprises of \( N \) stochastic processes \( \{\theta^i(t) : 1 \leq i \leq N\} \), where the value \( \theta^i(t) \in [0,2\pi] \) is the state of the
For large $N$, the empirical distribution of the population approximates the posterior distribution of the $\theta_i(t)$.

3. **Optimal control:** The turning objective is modeled as a partially observed optimal control problem. The optimal control law is shown to be a certain average taken over the population, $\{ \theta_i(t) : 1 \leq i \leq N \}$.

The methodology of this paper is a synthesis of several papers from our research group at the University of Illinois: the feedback particle filtering methodology for diffusion appears in [30], [29], [28]; its application to the problem of phase estimation is described in [24], [26]; a feedback particle filter-based approach to optimal control of partially observed diffusions is the subject of [21], [25]. In the present paper, we bring together these research threads to propose a CPG architecture for control of locomotion.

The paper is strongly influenced by the geometric phase literature in robotic locomotion. Informally, the geometric phase characterizes the net motion of a robot (e.g., $q_1$ in the model problem) achieved through cyclic changes in its shape variables (e.g., $x$). For coupled planar rigid bodies of the kind considered here, explicit formulas for geometric phase appear in Krishnaprasad [18]. The general theory can be found in Marsden [20] with a lucid introduction in Kelly and Murray [16]. The use of geometric phase for control of robots appears in Murray and Sastry [22], Hatton and Choset [12].

Closely related to geometric phase is the idea of mechanical rectification that goes back originally to the work of Brockett [2], [19] and more recently finds expression in several papers of Iwasaki and collaborators [1], [8], [14], [5]. The use of a phase variable in control of locomotory gaits appears in [10], [9].

The remainder of this paper is organized as follows: In Sec. II the modeling of the two-body system is described and in Sec. III, the corresponding control results are introduced. The conclusions with some directions for future work are discussed in Sec. IV.

II. **Problem Formulation**

Consider a system of two planar rigid bodies, the head body ($B_1$) and the tail body ($B_2$), connected by a single degree of freedom pin joint as shown in Fig. 2. The joint is assumed to be actuated by a motor with drive torque $\tau$, and in addition incorporates a linear torsional spring with coefficient $\kappa$ and viscous friction with coefficient $b$. For body $B_i$, $m_i$ denotes the body’s mass and $I_i$ denotes the moment of inertia about it’s center-of-mass. The span of the body $B_i$ – the distance between it’s center-of-mass and the pin joint – is denoted by $d_i$.

A. **Modeling**

**Configuration:** The instantaneous configuration of the system is denoted by $q = (q_1, q_2) \in \mathbb{T}^2$, the 2-torus. Here, $q_i$ is the absolute orientation of the frame rigidly affixed to body $B_i$, at its center-of-mass. The absolute angular velocity is denoted as $\omega = (\omega_1, \omega_2)$, where $\omega_i = \dot{q}_i$.

**Lagrangian:** Ignoring the viscous friction (setting $b=0$), the Lagrangian of the system is given by,

$$L = \frac{1}{2} \omega^T \begin{bmatrix} I_1 & \lambda \cos(q_1 - q_2) I_2 \\ \lambda \cos(q_1 - q_2) & I_2 \end{bmatrix} \omega - \frac{\kappa}{2} (q_1 - q_2)^2,$$

where $\dot{I}_i$ and $\lambda_i$ are defined in Table I.

**Conservation law:** The Lagrangian is invariant under rotations $(q_1, q_2) \rightarrow (q_1 + \psi, q_2 + \psi)$ where $\psi \in [0, 2\pi]$. This means that the total angular momentum is conserved, i.e.,

$$\sum_{i=1}^{2} \frac{\partial L}{\partial \dot{q}_i} = \sum_{i=1}^{2} (\dot{I}_i + \lambda \cos(q_1 - q_2)) \omega_i = \mu = \text{Constant}. \quad (2)$$

The conservation law suggests a choice of new coordinates $(q_1, x)$ where $x := q_1 - q_2$ represents the internal shape variable. The coordinate $q_1 \in \mathbb{T}^1$ is referred to as the group variable; cf., [16]. The latter choice reflects the fact that orientation $q_1$ describes the group of planar rigid body rotations (SO(2)).

For the remainder of this paper, the constant $\mu$ is assumed to be zero.

**Dynamic equations:** The dynamics for the system are described by a second order ODE for the internal variable and a first order ODE for the group variable. The latter is in fact simply the conservation law (2). Using the Lagrangian (1),

![Fig. 2. Schematic of the two-body system.](image-url)
the equations of motion are obtained as,
\[ \ddot{x} = \frac{1}{\Delta} \left[ -\lambda \sin(x) \frac{A_1 A_2}{A_1 + A_2} \dot{x}^2 + (A_1 + A_1)(\tau(t) - \kappa x - bx) \right], \]
\[ \dot{q}_1 = \frac{T_2 + \lambda \cos(x)}{I_1 + I_2 + 2\lambda \cos(x)} \ddot{x} = \tilde{f}(x, \dot{x}), \]
where \( bx \) in Eq (3) is the force due to the presence of viscous friction, and \( \tau \) is the motor torque. The parameters \( \Delta \) and \( A_i \) are defined in Table I.

In literature, the decomposition into the internal variable and the group variable is used to simplify the control problem: Using (3), the torque input \( \tau(t) \) is first used to obtain the trajectory \( x(t) \) for the internal variable; Subsequently, the conservation law (4) is used to solve for \( q_1(t) \), the trajectory for the group variable. The Step 2 is referred to as the reconstruction procedure. Note that the torque required to generate a particular trajectory in the internal variables is not needed for the reconstruction; cf., [16]. This allows one to work with the reduced order model (4), by specifying the trajectory of the internal variable directly.

**Control approach of the present paper:** An open loop periodic input is assumed for the torque actuation,
\[ \tau(t) = \tau_0 \sin(\omega_0 t), \]
where \( \omega_0 \) is a frequency and \( \tau_0 \) is the amplitude. There is no control objective associated with this torque input, except to have the internal variable \( x \) oscillate in a periodic manner.

The control input is introduced as an additional term in the righthand-side of the first order equation (4), expressed as
\[ q_1 = \tilde{f}(x, \dot{x}, u). \]

For the numerical results reported in Sec. III, the control input enters via change in span of the tail body \( B_2 \):
\[ d_2(t) = d_2(1 + u(t)), \]
where \( d_2 \) is the nominal span and \( u(t) \) represents a small time-dependent perturbation due to control.

**Assumption A1** The control input \( u(t) \) does not affect the internal dynamics.

In practice, this assumption may not strictly hold. However, the idea is to introduce small control perturbations – formalized through introducing a small parameter \( \varepsilon \) in the formulation of the optimal control problem below – such that the control has negligible effect on the internal dynamics.

**Control objective:** The control objective is to choose the control input \( u(t) \) to turn the head body \( B_1 \), either clockwise \( (q_1 \text{ increases with time}) \) or counterclockwise \( (q_1 \text{ decreases with time}) \).

For this purpose, one assume noisy observations of the form:
\[ dz(t) = \tilde{h}(x(t), \dot{x}(t)) \, dt + \sigma_w \, dw(t) \]
where \( z(t) \in \mathbb{R} \) denotes the observation process and \( w(t) \) is a standard Wiener process. The observation function \( \tilde{h} \) is assumed to be \( C^1 \) and \( \sigma_w \) denotes the standard deviation parameter for the noise process \( w(t) \).
The control problem of clockwise turn is modeled as an optimization problem:

$$\min_{u_{0:T}} \mathbb{E} \left[ (q_1(T) - q_1(0)) + \frac{1}{2\varepsilon} \int_0^T u(t)^2 \, dt \right],$$  \hspace{1cm} (8)

where $T := \frac{2\pi}{\omega_0}$ and $\varepsilon > 0$ is a small control penalty parameter. It is assumed throughout that the minimum is over all inputs adapted to $Z^*_t = \{ Z_s : s \leq t \}$, and that the minimum exists.

The problem for turning counterclockwise may be modeled similarly simply by switching the sign on the first term of the control objective in (8).

### B. Phase model

Consider the second order equation (3) for the internal variable $x$, forced by a periodic torque input (5). The following assumption is made concerning its solution:

**Assumption A2** The solution of (3), under periodic forcing (5), is given by an isolated asymptotically stable periodic orbit (limit cycle) with period $2\pi / \omega_0$.

Denote the limit cycle solution as $X_{LC}(\theta(t)) = (x(t), \dot{x}(t))$, where $\theta(t) = (\theta(0) + \omega t) \mod 2\pi$. Denote the set of points on limit cycle as $\mathcal{P} \subset \mathbb{R}^2$. The map $X_{LC} : [0, 2\pi) \to \mathcal{P}$ is invertible and denoted by $\tilde{\phi} : \mathcal{P} \to [0, 2\pi)$ defined by

$$\tilde{\phi}(x) = \theta \iff X_{LC}(\theta) = x.$$

The inverse map is extended to a small neighbourhood $\mathcal{N}$ of limit cycle $\mathcal{P}$ by using notion of asymptotic phase. Let $\Phi : \mathcal{N} \times \mathbb{R} \to \mathcal{N}$ be the flow map so $\Phi(x_0, t)$ is the solution to dynamics with initial condition $x_0 \in \mathcal{N}$. Then the asymptotic phase of $x_0$ is defined as the phase of the point $y_0 \in \mathcal{P}$ such that $\| \Phi(x_0, t) - \Phi(y_0, t) \| \to 0$ as $t \to \infty$.

In terms of the phase variable $\theta$, the first order dynamics (6) of the group variable $q(t)$ are expressed as

$$\dot{q}(t) = f\left( X_{LC}(\theta(t)), u(t) \right) = f(\theta(t), u(t)).$$  \hspace{1cm} (9)

Similarly define an observation function on the limit cycle as,

$$h(\theta) := \hat{h}(X_{LC}(\theta)).$$

This leads to the observation model, counterpart of (7), in terms of phase variable:

$$\dot{z}(t) = h(\theta(t)) \, dt + \sigma_w \, dw(t),$$

where $w(t)$ is a standard Wiener process and $\sigma_w$ is the standard deviation parameter for the measurement noise.

### C. Numerics

Fig. 3 depicts some of the simulation results including the torque input $\tau(t)$ in part (a), the resulting state trajectory $x(t)$ in part (b), and the observation process $y(t)$ in part (c). The trajectory is obtained via a numerical solution of the second order ODE model (3). The observation model is taken as $\tilde{h}(x, \dot{x}) = x$. The other simulation parameters are tabulated in Table I.

At each discrete time, the observation depicted in Fig. 3(c) is obtained as $y(t) := \delta z(t) / \delta t$, where $\delta t$ denotes the discrete time step, and $\delta z(t) = z(t + \delta t) - z(t)$. In numerical simulations, $\delta t = 0.01$ is chosen.

Fig. 3(d) depicts the resulting orientation $q_1(t)$ of the head body $B_1$. The orientation is obtained by numerically integrating the first order ODE (4). It is seen that while $q_1$ oscillates with frequency $\omega_0$, there is no net change in the orientation (over a cycle). The purpose of introducing the control input $u(t)$ is to influence a net change in the orientation, based on the noisy sensory output $y(t)$.

### III. CONTROL DESIGN

By integrating the first order equation (9),

$$q_1(T) - q_1(0) = \int_0^T f(\theta(t), u(t)) \, dt.$$

Substituting this in (8), the optimization problem is expressed as

$$\min_{u_{0:T}} \mathbb{E} \left[ \int_0^T \left( f(\theta(t), u(t)) + \frac{1}{2\varepsilon} u(t)^2 \right) \, dt \right].$$

By employing a straightforward calculus of variation argument, the optimal control $u^*(t)$ is the solution of the following fixed-point equation:

$$u^*(t) = -\varepsilon \mathbb{E} \left[ \frac{\partial f}{\partial u}(\theta(t), u^*(t)) \bigg| Z^*_t \right].$$  \hspace{1cm} (10)

The expectation on the right-hand-side is evaluated by constructing a feedback particle filter for the $\theta(t)$ process. The filter is comprised on $N$ stochastic processes $\{ \theta^i(t) : 1 \leq i \leq N \}$, where the value $\theta^i(t) \in [0, 2\pi)$ is the state of the $i^{th}$ particle (oscillator) at time $t$. The dynamics evolve according to,

$$d\theta^i(t) = \omega^i \, dt + K(\theta^i(t), t) \circ (dZ(t) - \frac{h(\theta^i(t)) + \hat{h}(t)}{2} \, dt),$$

where $\omega^i = \omega_0$ and $\hat{h}(t) := \mathbb{E}[h(\theta^i(t)) | Z^*_t]$. In a numerical implementation, we approximate $\hat{h} \approx N^{-1} \sum_{i=1}^N h(\theta^i(t)) =: \hat{h}^{(N)}$. The gain function $K$ is obtained via a solution of a certain boundary value problem; cf., [25].
In terms of these particles, the expectation in (10) is approximated as

\[ u^*(t) = -\varepsilon \frac{1}{N} \sum_{i=1}^{N} \frac{\partial f}{\partial u}(\theta'(t), u'(t)) \tag{11} \]

The fixed-point equation (10) or its particle approximation (11) may be difficult to solve in closed-form. In the asymptotic limit of small parameter \( \varepsilon \to 0 \), a solution may be obtained via a perturbation method. This is done by writing a Taylor series expansion of the function \( f \):

\[ f(\theta, u) = f_0(\theta) + f_1(\theta)u + f_2(\theta)u^2 + \ldots, \]

where \( f_1(\theta) = \frac{\partial f}{\partial u}(\theta, u) \Big|_{u=0} \). Using this series expansion, the solution of (11) is approximated as

\[ u^*(t) = -\varepsilon \frac{1}{N} \sum_{i=1}^{N} f_1(\theta'(t)). \]

A. Numerics

Fig. 4 depicts the limit cycle solution \( X_{LC}(\theta) \) obtained from a numerical solution of (3). For the purpose of implementing optimal control, the limit cycle solution is approximated using the functional form:

\[ X_{LC}(\theta) \approx (r \sin(\theta), r \cos(\theta)), \tag{12} \]

where \( r = 0.56 \) is numerically determined.

Using this form, \( h(\theta) = r \sin(\theta) \), and

\[ f(\theta, u) = \frac{\lambda}{1 + \lambda} \cos(r \sin(\theta))(1 + u) \]

For small values of the parameter \( r \), \( f_1(\theta) \) is approximated as

\[ f_1(\theta) = C \cos(\theta) + O(r^2) \]

where \( C = r^2 \frac{2I_1 + 2I_2}{[I_1 + I_2 + 2\lambda]} \) is a constant. In this case, the optimal control (11) is approximated as,

\[ u^*(t) \approx \varepsilon C \frac{1}{N} \sum_{i=1}^{N} \cos(\theta'(t)). \tag{13} \]

Two sets of simulation studies were carried out to investigate the performance of the filter and the control system. In Scenario 1, only the feedback particle filter is simulated without the optimal control input. In Scenario 2, the controlled system is simulated with the optimal control input (13). The results of these simulation studies are summarized next:

1. Estimation results: In Scenario 1, the feedback particle filter algorithm is simulated in open loop, i.e. with \( u(t) \equiv 0 \).
In the presence of periodic torque input (5), trajectory \((x, \dot{x})\) is a limit cycle in the phase space. The initial conditions of \(N = 1000\) particles \(\theta_i\) are drawn from a uniform distribution on \([0, 2\pi]\).

Fig. 5(a) summarizes some of the results of the numerical simulation. The initial transients due to the initial condition converge rapidly – as shown in part (a). A single cycle from a time-window after transients have converged is also depicted. The figure compares the sample path of the actual state \(\theta(t)\) (as a solid line) with \(\pm tw\) standard deviation bounds (shaded area). The density at two time instants is depicted in part (b). These results show that the filter synchronizes and is able to track the phase accurately.

Note that at each time instant \(t\), the bounds and the density \(p(\theta, t)\) shown here are all approximated from the ensemble \(\{\theta(t_i)\}_{i=1}^{10}\).

2. Control results: In Scenario 2, we apply the control input \(u^*(t)\) according to (13); the parameter values are identical to Scenario 1. The estimation results for Scenario 2 are qualitatively similar to Scenario 1: After a brief transient, the filter synchronizes and is able to track the phase accurately.

Fig. 6 summarizes some of the results of the numerical simulation. Once the filter synchronizes, the control input serves to slowly turn (rotate) the head body. The control input \(u(t)\) is depicted in part (a), and the position \(q_1(t)\) is depicted in part (b).

IV. Conclusion

In this paper, a coupled oscillators-based framework for control of locomotory gaits was proposed. The framework was illustrated with the aid of an example problem of turning a two-body planar system. In future work, the framework will be applied to control of multi-body robotic systems such as snake robots.

References


