Stability Analysis via Averaging for Nonlinear Systems with Delays

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Abstract—In this work we consider stability of time-varying systems with time delays via the averaging method. It is shown that if the strong averaged system of a system affected by a small positive parameter $\varepsilon$ admits a Lyapunov-Razumikhin function for the input-to-state stability (ISS) property, then the original system satisfies a semiglobal and practical ISS-like property as the small parameter $\varepsilon$ converges to zero.

I. INTRODUCTION

Time varying systems affected by a positive small parameter are an important class of nonlinear systems in stability analysis. A powerful tool in this context is the averaging method, see, for instance, [1], [3], [5], [6], [7], [10], [11], and references therein. The key idea of the averaging method is to convert the stability analysis for a time varying system to the analysis of a time invariant system that can be approximated in certain manners by the original system as the small parameter approaches 0 (see Definitions 3.1 and 3.4). The averaging technique was initially developed to handle asymptotic stability for systems not affected by disturbances. To deal with external disturbances, the authors of [6] introduced the notions of strong averaging and weak averaging. The main difference of the two types of averaged systems lies in the roles played by the external inputs in the approximation. For a system not affected by external inputs, the strong- and weak-averaged systems both become averaged systems used for the original system, see [11].

Despite a rich collection of work based on the averaging technique for systems without time delays, not much has been done for delayed systems. This note is devoted to the averaging technique for delayed systems. In particular, we will focus on the notion of strong averaged systems introduced in [6].

In stability analysis for delayed systems, the Lyapunov-Krasovskii functionals and Lyapunov-Razumikhin functions have been the main tools. The existence of different types of Lyapunov-Krasovskii functionals is essential (and in many cases equivalent) to the corresponding stability properties. However, it can be quite challenging to construct such a functional defined on an infinite dimensional space even for some very simple systems. Lyapunov-Razumikhin functions, on the other hand, is defined on a finite dimensional space which can be much less difficult to construct and to work with. More importantly, the work in [9] built a bridge between the classic Razumikhin theorem and a small-gain result in robust stability, which allows one to convert the task of stability analysis for delayed systems to the one of robust stability analysis for systems without delays, an area where vast tools and results have been developed (see [12] for some more discussions). In the current paper, we will follow the Razumikhin approach to extend the results related to strong averages obtained in [6] to delayed systems. To be more specific, we will show that if a strong average of a system admits an ISS-Lyapunov-Razumikhin function, then the original system is ISS in a semi-global and practical sense as the small parameter approaches to zero. Treating asymptotic stability for systems without inputs as a special case, we will show that if an averaging system admits a Lyapunov-Razumikhin function, then the semi-global and practical asymptotic stability holds for the original system, which extends the result in [11] to the case with time delays.

Notations. In this note, we will use $| \cdot |$ for the Euclidean norm of vectors, and $\| \cdot \|_I$ for the $L_\infty$ norm of measurable functions on the interval $I$, where $\| \cdot \|$ means the $L_{\infty}$ norm on $[0, \infty)$. For $\psi = (\psi_1, \ldots, \psi_k)$, we let $\| \psi \|_I = \max_{1 \leq i \leq k} \{ \| \psi_i \|_I \}$. A continuous function $\sigma : \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is of class $K$ if it is strictly increasing and $\sigma(0) = 0$; and is of class $K_\infty$ if it is of $K$ and unbounded. A function $\beta : \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0} \to \mathbb{R}_{\geq 0}$ is said to be of class $K\mathcal{L}$ if for each fixed $t \geq 0$, $\beta(\cdot, t)$ is of class $K$, and for each fixed $s \geq 0$, $\beta(s, t)$ decreases to 0 as $t \to \infty$. For any $K$-function $\kappa$, we say that $\kappa < \mathrm{id}$ if $\kappa(s) < s$ for all $s > 0$.

II. PRELIMINARIES

Let $\theta > 0$ be given. Consider the Banach space $X = C([-\theta, 0])$ of continuous functions from $[-\theta, 0]$ to $\mathbb{R}$, where for $\xi \in X$, the norm $\| \xi \|_X$ is defined to be the sup-norm of $\xi$. For a continuous function $v$ defined on $[-\theta, b]$, define $v_t \in X$ by $v_t(s) = v(t + s)$ for $s \in [-\theta, 0]$.

In this section we briefly review the Lyapunov-Razumikhin method for a general type of time-varying nonlinear systems affected by delays as follows:

$$\begin{align*}
\dot{x}(t) &= f(t, x(t), x_t, u(t)), \\
x(t) &= x_0(t), \quad t \in [t_0 - \theta, t_0],
\end{align*}$$

where the state variable $x(\cdot)$ takes values in $\mathbb{R}^n$, the input $u(\cdot)$ takes values in $\mathbb{R}^m$, and where $f : \mathbb{R}_{\geq 0} \times \mathbb{R}^n \times X^n \times \mathbb{R}^m$ is Lipschitz on compacts and maps bounded set to bounded set. Inputs $u(\cdot)$ are measurable and locally essentially bounded functions defined on $[0, \infty)$. Solutions of (1) are absolutely continuous functions that satisfies the equation almost everywhere. With the assumptions imposed on $f$, the usual regularity conditions such as existence,
uniqueness, and maximum continuation of solutions hold for (1).

The uniform input-to-state stability of (1) is defined in the same manner as in the delay-free case (c.f.[8], [4]). The system (1) is said to be uniformly input-to-state stable (uniformly iss) if for some \( \beta \in KL \) and some \( \gamma \in K \) the following holds for all trajectories of the system:

\[
|x(t)| \leq \max \left\{ \beta \left( \|x_o\|_{[t_0-\theta, t_0]}, t - t_0 \right), \gamma \left( \|u\|_{[t_0, \infty]} \right) \right\}
\]

for all \( t \geq t_0 \geq 0 \).

The underlying idea of the Razumikhin method is to treat the delayed state variables \( x_t \) in (1) as a disturbance in the system as in

\[
\dot{x}(t) = f(t, x(t), u(t)),
\]

where \( w_t \) is treated as an external disturbance. In this work the input \( u_t \) will be induced by absolutely continuous functions \( w \) in the same manner as \( x_t \). That is, each input \( w \) is an absolutely continuous function from \([\theta, \infty) \to \mathbb{R}^n \), and \( w_t \) is given accordingly by \( w_t(s) = w(t+s) \) for \( \theta \leq s \leq 0 \). This way, the delayed system (1) can be treated as the closed-loop system of (2) under the feedback \( v(t) = x(t) \). The next result, given in [9], provided a connection between the small-gain theorem and the Razumikhin theorem in the classic literature.

**Lemma 2.1:** Suppose that the system (1) admits a \( C^1 \) Lyapunov function \( R \times \mathbb{R}^n \to R \geq 0 \) such that the following hold:

- for some \( \alpha, \alpha \in K_\infty \), it holds that
  \[
  \alpha(|x|) \leq V(t, x) \leq \alpha(|x|) \quad \forall x;
  \]
- for some \( \kappa, \gamma \in K_\infty \), it holds that
  \[
  V(t, x) \geq \max \left\{ \kappa(\|V(s, w(s))\|_{[-\theta, 0]}, \gamma(|u|)) \right\} \quad (3)
  \]
  \[
  = D_t V(t, x) + D V(t, x) f(x, w_x, d) \leq -\alpha(V(x)).
  \]

If \( \kappa < \id \), then (1) is uniformly iss.

When the system (1) is time-invariant, \( V \) can be chosen to be time-invariant, and (3) is then reduced to

\[
V(x) = \min \left\{ \kappa(\|V(w(s))\|_{[-\theta, 0]}, \gamma(|u|)) \right\}
\]

\[
\Rightarrow D V(t, x) f(x, w_x, d) \leq -\alpha(V(x)).
\]

The time-invariant version of the next result was given in [12]:

**Lemma 2.2:** Assume that the trajectories of the system (2) associated with (1) satisfies an estimate with \( (w, u) \) as input:

\[
|x(t)| \leq \max \left\{ \beta(x(t_0), t - t_0) \right\},
\]

\[
\kappa \left( \|w\|_{[t_0-\theta, t_0]} \right), \gamma \left( \|u\|_{[t_0, \infty]} \right), \delta \right\}
\]

for all \( t \geq t_0 \geq 0 \). If \( \kappa < \id \), then for some \( \beta_1 \in KL, \rho \in K_\infty \) and \( \sigma \in K_\infty \), the following holds for trajectories of system (1):

\[
|x(t)| \leq \max \left\{ \beta_1(x(t_0), t - t_0), \rho(\|u\|_{[t_0, \infty]}), \sigma(\delta) \right\}
\]

for all \( t \geq 0 \).

**III. MAIN RESULTS**

Consider a time-varying system as follows:

\[
\dot{x}(t) = f(t, x(t), x_t, u(t)),
\]

where the state variable \( x(\cdot) \) takes values in \( \mathbb{R}^n \) input variable \( u(\cdot) \) takes values in \( \mathbb{R}^m \), and where \( f : \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^m \to \mathbb{R} \) satisfies the following:

(A1) \( f (t, x, v, u) : t \geq 0, (x, v, u) \in W \)

and some \( M > 0 \) such that

\[
\|f(t, x_1, v_1, u_1) - f(t, x_2, v_2, u_2)\| \\
\leq M \left( \|x_1 - x_2\| + \|v_1 - v_2\| + \|u_1 - u_2\| \right)
\]

for all \( (x, v, u) \in W \), for all \( t \geq 0 \).

(A2) \( f \) is completely continuous in \( (x, v, u) \) uniformly in \( t \), that is, for every bounded subset \( B \) of \( \mathbb{R}^n \times \mathbb{R}^m \), the set

\[
\left\{ f(t, x, v, u) : t \geq 0, (x, v, u) \in B \right\}
\]

is bounded.

Note that for a delay-free system, the map \( f \) is defined in the finite dimensional space \( \mathbb{R}^n \times \mathbb{R}^m \). Hence, in combination with condition (A1), the condition (A2) for the delay-free case can be reduced to the requirement that \( f(t, 0, 0) \) be bounded (which is a standard assumption in the literature, see e.g., [6] and [11]).

The input \( u \) of the system is a measurable and locally essentially bounded function on \( \mathbb{R} \geq 0 \). With the assumptions (A1)–(A2) imposed on \( f \), the existence, uniqueness, and maximum continuation of solutions hold for (4) (c.f. [2] and [12]). We will denote the maximum solution of (4) corresponding to the initial state \( \xi \) (i.e., \( x(s) = \xi(s) \) for \( s \in [t_0 - \theta, t_0] \)) and the input \( u \) by \( x(\cdot, t_0, \xi, u) \).

For a system as in (4), we consider the following system with a small parameter \( \epsilon > 0 \):

\[
\dot{x}(t) = f \left( \frac{t}{\epsilon}, x(t), x_t, u(t) \right).
\]

For a trajectory \( x(\cdot, \xi, u) \) defined on \([-\theta, T]\), let \( \varphi(\tau) = x(\epsilon \tau) \), then \( \varphi \) satisfies the following:

\[
\frac{d}{d\tau} \varphi(\tau) = \epsilon f(\tau, \varphi(\tau), T_\epsilon(\varphi, u_\epsilon)),
\]

where \( T_\epsilon \) maps from \( C[-\theta/\epsilon, 0] \) to \( C[-\theta, 0] \) by \( (T_\epsilon \psi)(s) = \psi(s/\epsilon) \), and \( u_\epsilon : [0, \infty) \to \mathbb{R}^m \) is defined by \( u_\epsilon(t) = u(\epsilon t) \).

It is seen that in the delay free case, (6) is the same type of systems studied in the context of averaging in [3].

**Definition 3.1:** A locally Lipschitz function \( f_{sa} : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}^n \) is said to be a strong average of \( f(t, x, w, u) \)

\[1\] Since \( \mathbb{R}^n \) is not locally compact, the condition of Lipschitz on compacts is weaker than the local Lipschitz condition which requires a function to be Lipschitz in a neighborhood of any given point.
if there exist $\beta_{av} \in KL$ and $T^* > 0$ such that for all $t > 0$, $w \in X^n, u \in L_\infty$, the following holds for all $T \geq T^*$:
\[
\frac{1}{T} \int_t^{t+T} \left( f_{sa}(x, w, u(s)) - f(s, x, w, u(s)) \right) ds \leq \beta_{av} \left( \max \{ |x|, \|w\|_\infty, \|u\|_\infty, 1 \} \right) T. \tag{7}
\]

Note that a function $f_{sa}$ satisfying (7) is in fact an average of $f$ of the strong type in $u$ and of the weak type in $w$ as defined in [6].

It is not always possible for a map $f$ to admit a strong average, see discussions in [6]. In the case when $f$ admits a strong average, the following system is called a strong average of (4):
\[
\dot{x}(t) = f_{sa}(x(t), x_t, u(t)). \tag{8}
\]

A $C^1$ function $V : \mathbb{R}^n \to \mathbb{R}_{\geq 0}$ is called an ISS-Lyapunov-Razumikhin function for the averaging system (8) if the following properties hold:
- for some $\alpha, \bar{\alpha} \in \mathcal{K}_\infty$,
  \[ \alpha(|x|) \leq V(x) \leq \bar{\alpha}(|x|), \tag{9} \]
- for some $\kappa, \rho \in \mathcal{K}$ and some $\varepsilon \in \mathcal{K}_\infty$, it holds that
  \[ V(x) \geq \max \{ \kappa(\|V(w)\|_{-\theta, 0}), \rho(|u|) \} \]
  \[ \Rightarrow \quad D_V(x)f_{sa}(x, w, u) \leq -\alpha(V(x)), \tag{10} \]
  \[ \kappa(s) < s \text{ for all } s > 0. \]

**Theorem 1:** Assume that the system (4) admits a strong average for which there exists a Lyapunov-Razumikhin function $V$ that satisfies (9)–(10) for some $\kappa < \text{id}$. Then there exists $\beta \in KL$ and $\rho \in \mathcal{K}$ such that for any given $R > 0$ and $\delta > 0$, there exists $\varepsilon^* > 0$ so that if $\varepsilon \in (0, \varepsilon^*)$, then for all trajectories of (5) that satisfies
\[
\max \{ \|x\|_{t_0-\theta, t_0}, \|\dot{x}\|_{t_0-\theta, t_0}, \|u\| \} \leq R, \tag{11}
\]
the following holds for all $t \geq 0$:
\[
|x(t)| \leq \max \{ \beta(\|x\|_{t_0-\theta, t_0}, t-t_0), \rho(\|u\|) \}, \delta \}. \tag{12}
\]

To utilize the Razumikhin approach, we associate with (5) the following system:
\[
\dot{x}(t) = f \left( \frac{t}{\varepsilon}, x(t), x_t, u(t) \right), \tag{13}
\]
where $w_t$ is treated as an external input induced by an absolutely continuous function $w(t)$. The following is one of the key results in the proof of Theorem 1.

**Proposition 3.2:** Assume that the system (8) admits a Lyapunov–Razumikhin function $V$ that satisfies (9)–(10) for some $\kappa < \text{id}$. Then there exists $\beta \in KL$ such that for any given $R > 0$ and $\delta > 0$, there exists $\varepsilon, \delta, \bar{\delta}, R > 0$ so that if $\varepsilon \in (0, \varepsilon^*, \bar{\delta}, R)$, then for all trajectories of (13) that satisfies
\[
\max \{ |x(t_0)|, \|\dot{x}\|_{t_0-\theta, t_0}, \|w\|_{t_0-\theta, t_0}, \|u\| \} \leq R, \tag{14}
\]
the following holds for all $t \geq 0$:
\[
|x(t)| \leq \max \{ \beta(|x(t_0)|, t-t_0), \kappa(\|w\|), \rho(\|u\|), \delta \}. \tag{15}
\]

The proofs of Theorem 1 and Proposition 3.2 will be given in Section IV.

A. An Example

Consider the system
\[
\dot{x}(t) = -x^3(t) + \cos^2 \left( \frac{t}{\varepsilon} \right) \int_{t-\theta}^{t} x^3(\sigma) d\sigma
\]
\[
+ \frac{tx(t)}{\varepsilon + t} u(t) \quad (t \geq 0). \tag{16}
\]
For this system, the map $f$ is defined by
\[
f(t, x, w, u) = -x^3 + \cos^2 t \int_{\theta}^{t} w^3(\sigma) d\sigma + \frac{xt}{1+t} u.
\]
Below we show that the map
\[
f_{sa}(x, w, u) = -x^3 + \frac{1}{2} \int_{\theta}^{t} w^3(s) ds + xu
\]
is an average of $f$ as defined in Definition 3.1. To see this, note that
\[
\frac{1}{T} \int_{t}^{t+T} \left( f_{sa}(x, w, u(s)) - f(s, x, w, u(s)) \right) ds
\]
\[
= \frac{1}{T} \int_{t}^{t+T} \left( \frac{1}{2} - \cos^2 s \right) \int_{\theta}^{t} w^3(\sigma) d\sigma + \frac{xu(s)}{1+s} ds
\]
\[
\leq \frac{1}{T} \int_{t}^{t+T} \left( \frac{1}{2} \int_{\theta}^{t} w^3(s) ds + \frac{xu(s)}{1+s} ds \right) \tag{17}
\]
\[
+ \frac{1}{T} |x| \|u\| \ln(1 + \frac{T}{1+t})
\]
\[
\leq \frac{1}{T} \left( \frac{1}{2} \|w\|^3_{\infty} + |x| \|u\| \ln(1 + T) \right)
\]
\[
\leq \beta \left( \max \{ |x|, \|w\|_{\infty}, \|u\|, 1 \}, T \right)
\]
for all $t \geq 0$, where $\beta(r, T) = r^2 \ln(1+T) + \frac{\theta r^3}{2T}$. 

For the averaging system
\[
\dot{x}(t) = -x^3(t) + \frac{1}{2} \int_{t-\theta}^{t} x^3(s) ds + xu(t),
\]
consider the function $V(x) = \frac{x^2}{2}$. Then
\[
D_V(x)f_{sa}(x, w, u) = -x^4 + \frac{x}{2} \int_{\theta}^{t} w^3(\sigma) d\sigma + x^2 u
\]
\[
\leq -x^4 + \frac{\theta |x|^2}{2} \|w^3\|_{\infty} + x^2 |u|.
\]
For any given $\theta > 0$, let $\kappa(s) = \theta^{1/3} s$. Then, whenever $V(x) \geq \kappa(\|V(w(\cdot))\|_{\infty})$, it holds that
\[
D_V(x)f_{sa}(x, w, u) \leq -\frac{1}{2} x^4 + x^2 u \leq -\frac{1}{4} x^4 + 4u^2.
\]
This shows that $V$ is an ISS-Lyapunov-Razumikhin function for the averaging system of (16) for all $\theta \leq 1$. Consequently, the system (16) satisfies the property of practical semi-global stability as stated in Theorem 1.

**Remark 1.3:** As discussed earlier, the function $f_{sa}$ is in fact an average of $f$ of the strong type in $u$ and of the weak type in $w$ (see [6] for detailed definitions). Indeed, similar to the argument used in [6], one can show that $f$ does not admit an average of the strong type in $w$. □
B. Systems without Inputs

Consider a delayed system not affected by inputs:

\[ \dot{x}(t) = f(t, x(t), x_t) \tag{17} \]

with initial value \( x(s) = x_0(s) \) for \( s \in [t_0 - \theta, t_0] \). Still assume that \( f \) satisfies Assumptions (A1)-(A2). The next definition is an extension of the averaging notion in [11] to systems with delays.

**Definition 3.4:** A locally Lipschitz function \( f_{av} : \mathbb{R}^n \times X^n \to \mathbb{R}^n \) is said to be an average of \( f(t, x, w) \) if there exists \( \beta_{av} \in KL \) and \( T^* > 0 \) such that \( \forall t > 0, w \in X^n \), the following holds for all \( T \geq T^* \):

\[ \left| \frac{1}{T} \int_t^{t+T} \left( f_{av}(x, w) - f(s, x, w) \right) ds \right| \leq \beta_{av} \left( \max \{ |x|, \|w\|_X, 1 \} \right) . \tag{18} \]

A \( C^1 \) function \( V : \mathbb{R}^n \to \mathbb{R}_{\geq 0} \) is called a Lyapunov-Razumikhin function for the averaging system

\[ \dot{x}(t) = f_{av}(x(t), x_t) \tag{19} \]

if the following properties hold:

- for some \( \eta, \nu \in K_{\infty} \), (9) holds;
- for some \( \kappa \in K \) and some \( \alpha \in K^\infty \), it holds that

\[ V(x) \geq \kappa (\|V(w)\|_{[-\theta,0]}) \quad \Rightarrow \quad DV(x)f_{av}(x, w) \leq -\alpha(V(x)) \tag{20} \]

- \( \kappa(s) < s \) for all \( s > 0 \).

The following corollary of Theorem 1 extends the result in [11] to delayed systems in a context of Razumikhin approach:

**Corollary 3.5:** Assume that the averaging system (17) admits an average for which there exists a Lyapunov-Razumikhin function \( V \) that satisfies (9) and (20) for some \( \kappa < \text{id} \). Then there exists \( \beta \in KL \) such that for any given \( R > 0 \) and \( \delta > 0 \), there exists \( \varepsilon^* > 0 \) so that if \( \varepsilon \in (0, \varepsilon^*) \), then for all trajectories of (17) that satisfies

\[ \max \{\|x\|_{[t_0-\theta,t_0]}, \|\dot{x}\|_{[t_0-\theta,t_0]} \} \leq R, \tag{21} \]

the following holds for all \( t \geq 0 \):

\[ |x(t, \xi, u)| \leq \max \{ \beta(\|\xi\|_{[t_0-\theta,t_0]}, t), \delta \} . \tag{22} \]

**IV. Proofs**

In this section, we briefly discuss steps and ideas of the proofs of our results. Some details are omitted due to the length restriction. Throughout the section, we will assume that all conditions of Theorem 1 hold.

A. Technical Lemmas

The proof of Theorem 1 will rely on the following results. **Lemma 4.1:** For any \( r_x > 0, r_w > 0, r_u > 0 \) given, there exist some \( d > 0 \) and \( M > 0 \) such that, for each \( \varepsilon > 0 \), for any \( t_0 > 0 \), the following holds along all trajectories of (13) with \( |x(t_0)| \leq r_x, \|w\| \leq r_w, |u| \leq r_u \):

\[ |x(t_1) - x(t_2)| \leq M |t_1 - t_2| \quad \forall t_1, t_2 \in [t_0, t_0 + d] . \tag{23} \]

The following is a consequence of the continuity of \( V \):

**Corollary 4.2:** For any \( r_x > 0, r_w > 0, r_u > 0 \), there exists \( d > 0 \) such that for any \( \varepsilon > 0 \) and any \( t_0 \geq 0 \), if \( V(x(t_0)) \leq r_x, \|V(w(s))\|_{[t_0-\theta,t_0]} \leq r_w \) and \( \|u\| \leq r_u \), then the solution \( x(t) \) of (13) is defined on \( [t_0, t_0 + d] \) and

\[ |V(x(t_1)) - V(x(t_2))| \leq M |t_1 - t_2| \]

for all \( t_1, t_2 \in [t_0, t_0 + d] \).

**Lemma 4.3:** For any given \( \Delta > 0 \) and \( \delta > 0 \), there exists \( d^* > 0 \) such that for any \( d \in (0, d^*) \) there is \( \varepsilon_d > 0 \) so that for all \( \varepsilon \in (0, \varepsilon_d) \) and for all \( t_0 \geq 0 \), if

\[ \max \{ |x(t_0)|, \|w\|_{[t_0-\theta,t_0]}, \|\dot{w}\|_{[t_0-\theta,t_0]}, \|u\| \} \leq \Delta \] (24)

then the solution \( x(t) \) of (13) exists for all \( [t_0, t_0 + d] \) and satisfies the following:

\[ V(x(t)) \geq \max \{ \kappa(\|V(w)\|_X), \rho(\|u\|) \} + \delta \tag{25} \]

\[ \Rightarrow \quad V(x(t_0 + d)) - V(x(t_0)) \leq -\frac{\delta}{d} \alpha_3(V(x(t_0))) . \]

**Lemma 4.4:** The following holds for some \( \beta_V \in KL \): for any \( \Delta > 0 \) and \( \delta > 0 \), there exists \( d^* > 0 \) such that for any \( d \in (0, d^*) \), there is \( \varepsilon_d > 0 \) such that if a trajectory \( x(t) \) of (13) with \( \varepsilon < \varepsilon_d \) satisfies (24), then \( x(t) \) is defined for all \( t \geq t_0 \) and the following holds for \( V_k := V(x(t_0 + kd)) \) for all \( k \geq 0 \):

\[ V_k \leq \max \{ \beta_V(0, kd), \kappa(\|V(w)\|_{[-\theta,d]}), \rho(\|u\|) \} + \delta . \tag{26} \]

**Lemma 4.5:** There exists \( \beta_V \in KL \) such that for any given \( \Delta > \delta > 0 \), there is some \( \varepsilon^* > 0 \) so that for every \( \varepsilon < \varepsilon^* \) and every trajectory of (13) satisfying (14), it holds that

\[ V(x(t)) \leq \max \{ \beta_V(x(t_0), t-t_0), \kappa(\|V(w)\|), \rho(\|u\|) \} + \delta \]

for all \( t \geq t_0 \).

B. Proof of Lemma 4.3

Let \( \Delta > 0 \) and \( \delta > 0 \) be given, and let \( V \) be the Lyapunov-Razumikhin function of (8). By Corollary 4.2, there is some \( d_1 > 0 \) such that for every trajectory of (13) that satisfies (24), the following holds for all \( t \in [t_0, t_0 + d_1] \):

\[ |V(x(t)) - V(x(t_0))| \leq \frac{\delta}{4} \tag{26} \]

and \( |x(t)| \leq 2\Delta \) for all \( t \in [t_0, t_0 + d_1] \).

Consider the set

\[ \Omega_{\Delta} = \{ (x, w_t, u) : |x| \leq 2\Delta, \|w\|_{[t_0-\theta,t_0+d_1]} \leq \Delta, \|\dot{w}\|_{[t_0-\theta,t_0+d_1]} \leq \Delta, |u| \leq \Delta, t \in [t_0, t_0 + d_1] \} . \]
Then $\Omega_\Delta$ is compact. Let $L > 0$ be the uniform Lipschitz constant for $DV(x), f(t/\varepsilon, x, w_t, u), f_{sa}(x, w_t, u)$ over $\Omega_\Delta$. Let $T^*$ be large enough so that

$$\beta(\max\{\Delta, 1\}, T) \leq \frac{\alpha(\delta/2)}{4L\Delta}$$

for all $T \geq T^*$. Let $d_2$ be small enough that

$$2Md_2 + 2\alpha^{-1}(\alpha(\delta/4) + 4L^2 M d_2 + 4KLM d_2) \leq \delta,$$

where $M$ is as in Corollary 4.2. In particular, $M d_2 < \delta/2$.

Define $d^* := \min\{d_1, d_2\}$ and let $d \in (0, d^*)$, and $d_\delta > 0$ such that $d/d_\delta > T^*$. Pick a trajectory of (13) that satisfies (24). Assume that

$$V(x(t_0)) \geq \max\{\kappa(||V(w)||), \rho(||u||)\} + \frac{\delta}{2}. \quad (27)$$

It follows from (26) that

$$V(x(t)) \geq \max\{\kappa(||V(w)||), \rho(||u||)\}$$

on $[0, d^*]$. It can be shown that for all $d \leq d^*$,

$$V(x(t_0 + d)) - V(x(t_0)) \leq -d\alpha_\Delta(x(t_0)/2) + \frac{d}{4}\alpha(\delta/2) + d^2 K_\Delta,$$

where $K_\Delta > 0$ is a constant depending only on $\Delta$. Hence, by properly choosing $d^*$, one concludes that if (27) holds on $[0, d]$ and if $\varepsilon < d_\delta$, then

$$V(x(t_0 + d)) - V(x(t_0)) \leq -\frac{d}{2}\alpha(0.5V(x(t_0)))$$

for all $d \leq d^*$.

**C. Proof of Lemma 4.4**

Without loss of generality, we assume that $\Delta \geq 2\delta$, and let

$$\Delta_1 = \max\{\kappa(\Delta), \rho(\Delta)\} + \delta/2.$$

Let $d^*$ be as in Lemma 4.3 chosen based on $(\Delta_1, \delta)$. Also assume that $d^*$ is small enough that (26) holds on $[t_0, t_0 + d^*]$ for all trajectories satisfying

$$|x(t_0)| \leq \Delta_1,$$

$$\max\{|u||_{t_0 - \theta, t_0 + d}, |u||_{t_0 - \theta, t_0 + d}, |u|| \} \leq \Delta_1. \quad (28)$$

Let $x(t)$ be a trajectory of (13) with initial state $x_0$ and input $(w, u)$ satisfying (24), and let $[0, T]$ denote the maximum interval of $x(t)$. Let $x_k := x(t_0 + kd)$, where $kd < T$.

**Claim.** Let $T_0 < T$ be given. If $V(x_0) \leq \Delta_1$, then $V(x_k) \leq \Delta_1$ for all $k$ with $kd < T_0$.

**Proof.** It is enough to prove this for $k = 1$. For any $t \in [t_0, t_0 + d]$, if $V(x(t)) < \delta$, then by Corollary 4.2,

$$V(x(t)) \leq V(x(t_0)) + \delta \leq 2\delta \leq \Delta.$$

Assume $V(x(t_0)) \geq \delta$, then by (26), it holds that

$$V(x(t)) \geq \max\{\kappa(||V(w)||), \rho(||u||)\}$$

for all $t \in [t_0, t_0 + d]$. By Lemma 4.3,

$$V(x(t)) - V(x(t_0)) \leq -\frac{d}{2}\alpha(0.5V(x(t_0))) \leq 0.$$

This implies that

$$V(x(t)) \leq \max\{V(x(t_0)), \kappa(||V(w)||), \rho(||u||)\} \leq \Delta_1.$$

Thus, in both cases whether $V(x(t_0)) \leq \delta$, one has

$$V(x(t)) \leq \max\{\Delta_1, \kappa(||V(w)||), \rho(||u||)\} = \Delta_1$$

for all $t \in [t_0, t_0 + d]$. The claim is thus proved.

Note that in the above proof, $k$ is chosen so that $T_0 - kd < \delta$, and hence,

$$V(x(t)) \leq V(x_k) + \delta/2 \quad \forall t \in [t_k, T_0].$$

Thus, $V(x(t)) \leq \Delta_1 + \delta/2$ for all $t \in [t_0, T]$. As a consequence, $T = \infty$, i.e., $x(t)$ is defined on $[t_0, \infty]$.

For $s \geq t_0$, define $p(s)$ by

$$p(s) := V(x_k) + \left(\frac{s}{d} - k\right)(V(x_{k+1}) - V(x_k))$$

on $[kd, (k+1)d]$, $k \geq 0$. Observe that $p$ is a linear interpolation of $V(x_k)$ and $V(x_{k+1})$ for $s \in [kd, (k+1)d]$. So, $p(s) \leq \Delta_1$ for all $s \geq 0$, and

$$0 \leq p(s) \leq \max\{V(x_k), V(x_{k+1})\}$$

for all $s \in [kd, (k+1)d], k \geq 0$.

It follows from (26) that

$$p(s) \leq V(x_k) + \frac{\delta}{4}$$

on $[kd, (k+1)d]$. Note that

$$p(s) \geq \max\{\kappa(||V(w)||), \rho(||u||)\} + \frac{\delta}{2}$$

$$\implies V(x_k) \geq \max\{\kappa(||V(w)||), \rho(||u||)\} + \frac{\delta}{4}.$$  

$$\implies V(x_{k+1}) - V(x_k) \leq -\frac{d}{2}\alpha(0.5V(x_k)).$$

By the definition of $p(s)$, we have

$$p(s) \geq \max\{\kappa(||V(w)||), \rho(||u||)\} + \frac{\delta}{2}$$

$$\implies \frac{d}{ds}p(s) \leq -\frac{1}{2}\alpha(0.5V(x_k)).$$

It can be further deduced that

$$p(s) \geq \max\{\kappa(||V(w)||), \rho(||u||)\} + \frac{\delta}{2}$$

$$\implies \frac{d}{ds}p(s) \leq -\frac{1}{2}\alpha(0.5p(s)).$$

Therefore, there exists some $\beta_\varepsilon \in KL$ (which depends only on $\alpha$) such that

$$p(s) \leq \max\{\beta_\varepsilon(p(0), s), \kappa(||V(w)||), \rho(||u||)\} + \frac{\delta}{2}$$

for all $s \geq 0$. As a consequence, for all $k \geq 1,

$$V(x_k) \leq \max\{\beta_\varepsilon(V(x_0), kd), \kappa(||V(w)||), \rho(||u||)\} + \frac{\delta}{2}. \quad (29)$$
D. Proof of Lemma 4.5

For a given $\Delta > 0$, let $\Delta_1$ be defined as in the proof of Lemma 4.4. Let $d^\ast$ be as in Lemma 4.4 for $\Delta$, and $d_1$ as in Lemma 4.4 based on $d$, $\Delta_1$, and $\delta/2$. Then for any trajectory of (13) with $\varepsilon < \varepsilon_3\Delta$, (29) and (26) holds.

The for any $t \in [t_k, t_{k+1}]$,

$$V(x(t)) \leq V(x(t_k)) + \delta/2.$$  

Define $\beta_\alpha$ by $\beta_\alpha(s, t) := 2 \max_{\eta \in [t_k, t]} 2^{-\eta} \beta_\alpha(s - \eta)$. Note that

$$\beta_\alpha(s, (k+1)d) = 2 \max_{\eta \in [0, (k+1)d]} 2^{-\eta} \beta_\alpha(s - \eta) \geq 2 \cdot 2^{-d} \beta_\alpha(s, kd) \geq \beta_\alpha(s, kd)$$

and since $t - t_0 < (k+1)d$ for all $t \in [t_k, t_{k+1}]$, $k \geq 0$, we have

$$V(x(t)) \leq V(x(t_k)) + \delta/2$$

for all $t \in [t_0, t_0 + T]$.

Observe that Proposition 3.2 follows from Lemma 4.5 immediately.

E. Proof of Theorem 1

Let $0 < \delta < R$ be given. By Proposition 3.2, (15) holds for all trajectories of (13) if $\varepsilon < \varepsilon_3R$ and if (14) holds. Observe that with $w = x(t)$, it follows that $\varepsilon < \varepsilon_3R$, then for all trajectories of (5) satisfying

$$\max\{\|x\|_{[t_0, t]}, \|\dot{x}\|_{[t_0, t]}, \|u\|\} \leq R,$$  

the following holds on the maximum interval of $x(\cdot)$:

$$|x(t)| \leq \max\{\beta(\|x(t_0\|, t - t_0), \kappa(\|V(w)\|), \rho(\|u\|)), \delta\}. $$  

Define $\Gamma_0(R) = 2 \max\{\beta(0, 0), R, \rho(R)\}$, and let

$$M_R = \max\{|f(s, x, w, u)| : s \geq 0, \max\{\|x\|, \|u\|, \|u\|\} \leq \Gamma_0(R)\}. $$

Let $\Gamma(R) = \max\{\beta(R, 0), R, M_R, \rho(R)\}$. Consider a trajectory of (5) with $\varepsilon < \varepsilon_3\Gamma(R)$. Consider a trajectory of (5) with $\varepsilon < \varepsilon_3\Gamma(R)$.

Claim. If (11) holds, then for all $t \geq t_0$,

$$|x(t)| \leq \Gamma_0(R), \|\dot{x}\| \leq \Gamma(R).$$  

Proof. Note that on any interval $[0, b]$ where $|x(t)| \leq \Gamma_0(R)$, it holds that $|\dot{x}(t)| \leq \Gamma(R)$.

Suppose (32) fails for some $t < T$, where $[0, T)$ is a maximum interval of the trajectory. Let

$$t_1 = \inf\{t : |x(t)| \geq \Gamma_0(R)\}. $$

Then, on $[0, t_1)$, it holds

$$\max\{\|\dot{x}\|_{[t_0, t]}, \|x\|_{[t_0, t]}, \|u\|\} \leq \Gamma_0(R).$$

As a consequence,

$$|x(t)| \leq \max\{\beta(R, 0), \kappa(R), \rho(R), \delta\},$$

for all $t \in [t_0 - \theta, t_1)$. By continuity, $|x(t_1)| \leq \Gamma_0(R)/2 < \Gamma_0(R)$, contradicting the definition of $t_1$. This proves the claim.

We have shown that if $\varepsilon < \varepsilon_3\Gamma(R)$, then condition (11) implies that

$$\max\{\|x\|_{[t_0, t]}, \|\dot{x}\|_{[t_0, t]}, \|u\|\} \leq \Gamma(R),$$

which, in turn, implies (31) for all $t \geq t_0$. Combining this with Lemma 2.2, we complete the proof.

V. Conclusions

We studied the ISS property for time-varying nonlinear delayed systems affected by a small positive parameter, and we extended the results on strong averaging systems obtained in [6] to systems with delays. The extension was based on the Lyapunov-Razumikhin approach, which allows one to apply robust stability analysis for delay-free systems to systems with delays without using the Lyapunov-Krasovskii functionals which are defined on infinite dimensional spaces. A critical step lies on getting rid of assumptions on $\dot{x}$ along the whole trajectory. By using similar approaches, it is very likely the results on weak averaging systems can be extended to systems affected by delays.

REFERENCES