Stochastic Reachability for Control of Spacecraft Relative Motion*

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Abstract—The concept of stochastic reachability allows for the assessment, before any maneuvers are initiated, of the probability of successfully implementing a rendezvous or docking procedure for spacecraft. The so-called reach-avoid problem lets us find the probability of reaching a target set while avoiding some unsafe or undesired set, despite uncertainty due to nonlinearity and disturbances. This paper examines two novel methods for the calculation of stochastic reachable sets, and specifically for rendezvous and docking problems. In particular, we examine a) particle (or scenario) approximations to expected values, and b) conversion of the reach-avoid probability to a chance-constrained convex optimization problem. Both methods allow for computation of the reach-avoid set in higher dimensions, as compared to other existing methods for computing stochastic reachable sets. We describe in detail both of these methods, and then apply them to spacecraft relative motion, a four-dimensional problem.

I. INTRODUCTION

The ability to safely perform spacecraft rendezvous and docking maneuvers is crucial in many applications, for instance in repairing satellites, resupplying the space station, and other missions. While there is extensive literature on control mechanisms for performing such maneuvers, particularly the use of model predictive control with approach and docking constraints for successful rendezvous (see, e.g., [1], [2], [3], [4]), almost no work has been done on characterizing the initial set of states from which such maneuvers can be performed safely.

As with many other cyber-physical systems, the tight integration of nontrivial dynamics and hierarchical mode logic makes verification paramount. The potential for damage or loss in such expensive systems, in combination with prohibitively long communication delay, means that autonomy must be accurate, reliable, and effective despite uncertainties in modeling and in disturbance forces. For spacecraft dynamics, there are many initial states from which no control input will lead to a safe or desirable outcome (i.e. successful docking or rendezvous). The risk of costly failures can be reduced by determining whether a rendezvous-type maneuver is initialized from a position where success is guaranteed, or is guaranteed with high probability.

Reachability analysis is a useful tool for identifying the set of initial states in a dynamical system that can be driven to a desired target set, to avoid some undesired set of states, or to accomplish both (the “reach-avoid” problem). There has been significant work on reachability and on calculating reachable sets, in both stochastic and deterministic, and linear and non-linear settings. In this paper we focus on stochastic reachability: With uncertainty in the dynamics, any type of reachable set becomes probabilistic. In this case we do not have hard guarantees of reaching a desired state, but rather seek the set of initial states that will reach a target with highest probability.

Recently there has been a great deal of work on stochastic reach and reach-avoid problems in both continuous and discrete time. In continuous time, level set methods can be used to approximate solutions to a stochastic Hamilton-Jacobi-Bellman equation, as in [5], [6]. In discrete time, dynamic programming has been used to solve such problems (and in particular for hybrid system dynamics, as in [7], [8]). In both cases, as the dimensionality of the system grows, the computational effort required for such problems renders a solution unattainable. Current efforts have been limited to at most three dimensional systems.

The linearized discrete-time Clohessy - Wiltshire - Hill (CWH) equations for spacecraft relative motion with added stochastic noise can be broken up into in-plane and out-of-plane motion, i.e. a four-dimensional problem in $x$, $y$, $\dot{x}$, $\dot{y}$ and a separate two-dimensional problem in $z$ and $\dot{z}$, within the Hill reference frame. Unfortunately, even the four-dimensional problem is beyond the limits of standard dynamical programming, which to our knowledge is the only established method for solving discrete time stochastic reachability problems (without assuming a worst-case bounded disturbance added to linear dynamics, as in [9]). This has led us to explore alternative methods for calculating reach sets which are more tractable for larger-scale problems.

Specifically, we have drawn upon the existing literature for solving stochastic control problems with chance constraints in order to develop novel techniques for calculating stochastic reach-avoid sets for higher dimensional systems, and applied these techniques to a realistic four dimensional system.

A chance constrained optimal control problem requires minimizing some objective (such as minimizing the control effort) while enforcing a probabilistic constraint, such as remaining within a safe region $K$ with some desired probability. We examine two approaches: The first is based
on sampling from the distribution of the noise to approximate expected values and probabilities (referred to as a “particle” or “scenario” approximation, see [10]), and was used in chance-constrained predictive control in [11] amongst others. The second approach is based on a convex under-approximation of the probability of remaining within a set \( K \), assuming the additive noise is Gaussian, that allows the reachability problem to be solved using convex optimization methods. This method was presented in [12] and used again in [13], and applied to systems in two dimensions. In both cases, we exploit the linearity of the dynamics when applying them to the reachability problem.

After presenting the reachability problem for spacecraft rendezvous and docking in Section II, we show how the problem can be solved using an open-loop controller first using sampling to approximate the reach-avoid probability, which is maximized using a mixed-integer linear programming (MILP) algorithm, in Section III-A. Next, we reformulate the maximal reach-avoid probability as a convex chance-constrained optimization problem in Section III-B. In Section III-C, we reformulate the approach of Section III-A to allow for a state-based feedback controller. In Section IV we compute the set of initial states that lead to successful rendezvous with highest probability using all of the above methods. Finally, we discuss possible further work and extensions to our current results in Section V.

II. PROBLEM FORMULATION

A. Relative Motion Equations

We consider the problem of a spacecraft (the deputy) approaching some target (the chief), with the in-plane dynamics of the approaching spacecraft relative to the target given by the linearized time-invariant CWH equations [14]:

\[
\begin{align*}
\dot{x} - 3\omega^2 x - 2\omega y &= \frac{F_x}{m_c} \tag{1} \\
\dot{y} + 2\omega x &= \frac{F_y}{m_c} \tag{2}
\end{align*}
\]

Here \( F_x \) and \( F_y \) are the components of the external force vector (i.e. the thruster control input), \( m_c \) is the mass of the deputy, and \( \omega = \sqrt{\frac{\mu}{R_0^3}} \) where \( \mu \) is the gravitational constant and \( R_0 \) is the orbital radius of the spacecraft. In the following we will only consider the in-plane motion since the out-of-plane motion is decoupled from the \( x, y \) dynamics. We then discretize (1)-(2) using time-step \( \Delta \) according to [15], and let \( X = [x, y, \dot{x}, \dot{y}]^T, \ u = [F_x, F_y]^T \). We obtain discretized dynamics of the form

\[
X_{k+1} = AX_k + B u_k + v_k \tag{3}
\]

with added process noise vector \( v_k \in \mathbb{R}^4 \), that represents uncertainty in the model due to external forces on the spacecraft not captured in the linearized model. For simplicity, we assume this noise is zero-mean gaussian, with covariance matrix \( \Sigma \).

B. Reachability

We next examine the problem of controlling the spacecraft to approach the chief according to the dynamics defined above. For a docking problem, there are many safety constraints that must be satisfied for the operation to be successful. For instance, the spacecraft should come close to the chief without hitting it, while staying within a line-of-sight (LoS) cone relative to the chief. Staying within an LoS cone is necessary when there are sensors on the chief measuring the location of the approaching spacecraft. While we do not incorporate sensor measurements, it is a natural next step, and so we assume all necessary requirements as if sensor measurements were taken.

We define the LoS cone as the set \( K \), which is the set we want to remain inside (i.e. we wish to avoid the complement of \( K \)) by requiring \( |x_k| \leq |y_k| \forall k \). Target set \( T \) is a small box close to the chief that the deputy should reach at terminal time \( N \), defined by \( |x_N| \leq \epsilon_x, \epsilon_y \leq y_N \leq \epsilon_y \leq 0 \), with \( \epsilon_x, \epsilon_y \) sufficiently small to represent successful docking (i.e. \((x_N, y_N)\) is almost at the origin). The set \( T \) also includes bounds on the velocity components \( \dot{x}_N, \dot{y}_N \) so that the deputy does not crash or dock with excess force.

For a given initial state \( X_0 \in K \) and set of control inputs \( U = [u_0, \ldots, u_{N-1}]^T \), the probability of staying within set \( K \) for times \( 1, \ldots, N-1 \) and reaching set \( T \) at time \( N \) is defined as:

\[
P_{X_0}^{U,N}(K, T) = P[X_1, \ldots, X_{N-1} \in K, X_N \in T \mid X_0, U] \tag{4}
\]

We would like to maximize the probability, starting from state \( X_0 \), of staying within \( K \) and reaching \( T \), so we define

\[
P_{X_0}^N(K, T) = \max_{U \in \mathcal{U}} P_{X_0}^{U,N}(K, T) \tag{5}
\]

as the maximum probability over all possible controllers given by the set \( \mathcal{U} \). We first consider open loop controllers bounded between some maximum and minimum value \( (u_{\min} \leq u_k \leq u_{\max}, k = 0, \ldots, N-1) \) because they are the simplest to implement. We later consider a feedback controller of the form \( u_k = \sum_{i=0}^{k} K_i x_i + r \) for the particle approximation method in Section III-C.

Equation (4) can be rewritten as in [7], [8] using indicator notation, with \( 1_A(x) = 1 \) for \( x \in A \) and \( 1_A(x) = 0 \) otherwise. Since \( P[x \in A] = \mathbb{E}[1_A(x)] \), with \( \mathbb{E} \) denoting expected value, the problem can be restated in optimal control notation:

\[
P_{X_0}^{U,N} = \mathbb{E}_X X_0 \left( \prod_{i=1}^{N-1} 1_{K}(X_i) \right) 1_T(X_N) \tag{6}
\]

with

\[
\left( \prod_{i=1}^{N-1} 1_{K}(X_i) \right) 1_T(X_N) = \begin{cases} 1 \text{ if } X_1, \ldots, X_{N-1} \in K \\ 0 \text{ else} \end{cases}
\]

Finally, the reach-avoid set for some desired probability \( 1 - \alpha \) is given by

\[
R_{\alpha}^N(K, T) = \{ X_0 : P_{X_0}^{N}(K, T) \geq 1 - \alpha \} \tag{7}
\]
Therefore, calculating the reachable set reduces to solving the optimal control problem (5) over all possible initial states $X_0 \in K$, and discovering for which $X_0$ the reach-avoid probability is at least $1 - \alpha$.

The set $T$ is partially contained in $K$, so requiring the deputy to stay within $K$ until time $N - 1$ may lead to the deputy being within $T$ before time $N$. The requirement however is that $X_N \in T$ at exactly time $N$, even if $T$ was reached beforehand. The more generic problem of reaching $T$ at any time less than or equal to $N$ (at which point the maneuver is complete) could be defined by the union over all $n \leq N$ of the reach-avoid sets with $X_n \in T$, i.e.

$$RA_n(K,T) = \bigcup_{n=0}^{N} RA_n(K,T) \quad (8)$$

Calculation of the set of initial states that could reach $T$ at time $n \leq N$ and stay inside $K$ for all $k = 0, \ldots, n - 1$ therefore reduces to solving (7) for multiple values of $N$, and so (7) is the focus of this paper.

III. CALCULATING $p_{X_0}^N(K,T)$

The linearity of the system allows us to rewrite the dynamics in vector form, letting $X = [X_1, X_2, \cdots, X_N]^T$, $U = [u_0, u_1, \cdots, u_{N-1}]^T$, and $V = [v_0, v_1, \cdots, v_{N-1}]^T$. Then

$$X = \bar{X}_0 + HU + GV \quad (9)$$

with $\bar{X}_0$, $H$, and $G$ composed of combinations of the matrices $A$ and $B$ from (3) and vector $X_0$ (see [16]). Expressions for $\bar{X}_0$, $H$, and $G$ are derived by repeatedly solving (3) at each time $k$, using the previous expression for $X_{k-1}$ as a function of $A$, $B$, and $X_0$ to get an updated expression for $X_k$.

The problem of finding $p_{X_0}^N(K,T)$ can be written as a constrained stochastic optimization problem.

Problem 1 (Stochastic Optimal Control Problem)

$$\max \mathbb{E}_{X_0} \left[ \left( \prod_{i=1}^{N-1} 1_K(X_i) \right) 1_T(X_N) \right]$$

Subject to:

$$X = \bar{X}_0 + HU + GV$$

$$|u_i| \leq u_{\text{max}} \ \forall \ i = 0, \ldots, N - 1$$

A. Particle Approximation

When taking the expected value of a function of a random variable $f(x)$ with respect to some distribution $p(x)$, it is necessary to evaluate an integral of the form $E[f(x)] = \int f(x)p(x) \, dx$. However, this integral is often difficult or impossible to solve (as is the case for (6), where the dimension of the integral grows quite large). Instead, it is possible to approximate the integral by drawing independent, identically distributed random samples (particles) $x^{(1)}, x^{(2)}, \cdots$ from a proposal distribution $q(x)$ and calculating a weighted sample mean

$$\hat{E}[f(x)] = \frac{1}{M} \sum_{i=1}^{M} w_i f(x^{(i)}) \quad (10)$$

with $w_i = \frac{q(x^{(i)})}{q(x)}$ and $q(x)$ chosen so that $q(x) > 0$ whenever $p(x) > 0$. By the strong law of large numbers (and so making some weak assumptions on the boundedness of $f(x)$ and the moments of $p(x)$), it follows that $\hat{E}[f(x)] \to E[f(x)]$ as $M \to \infty$.

Because we can easily sample from the multivariate Gaussian distribution, we let $q(x) = p(x)$, where $p(x)$ is Gaussian, and so $w_i = 1$. We therefore draw $M$ samples $V^{(1)}, V^{(2)}, \ldots, V^{(M)}$, where $V^{(i)} \in \mathbb{R}^{4 \times N}$

$$V^{(i)} \sim \mathcal{N}(0, \mathcal{V}), \ \mathcal{V} = \begin{bmatrix} \Sigma & 0 & \cdots & 0 \\ 0 & \Sigma & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & \Sigma \end{bmatrix}$$

We then have $M$ realizations of the dynamics, given by

$$X^{(i)} = \bar{X}_0 + HU + GV^{(i)}$$

Then $p_{X_0}(K,T) \approx \frac{1}{M} \sum_{i=1}^{M} z_i$. Next, if the sets $K$ and $T$ are assumed convex, we may exploit the property that convex sets can be represented by a finite intersection of hyperplanes (see [11]), i.e.

$$X_{1:N-1} \in K \land X_N \in T \implies X \in \bigcap_{i=1}^{M} \{ X : a_l^T X \leq b_l \} \quad (12)$$

To convert Problem 1 to a mixed integer linear program, as in [11], we enforce constraints of the form (12) by using $z_i$ and (11), and some large number $C$

$$a_l^T X^{(i)} - b_l \leq C(1 - z_i) \ \forall \ i = 1, \ldots, M, \ l = 1, \ldots, M_l$$

so that for appropriately defined $a_l$, $b_l$ representing sets $K$ and $T$, and $C$ large enough, we force $z_i = 0$ when $X^{(i)}_n \notin K$ for some $1 \leq n \leq N - 1$, or when $X^{(i)}_N \notin T$. We want to find an open loop control $U$ that produces as many $z_i$ equal to 1 as possible.

Problem 2 (Particle Approximation to Problem 1)

$$\max \sum_{i=1}^{M} z_i$$

Subject to:

$$X^{(i)} = \bar{X}_0 + HU + GV^{(i)} \ \forall \ i = 1, \ldots, M$$

$$a_l^T X^{(i)} - b_l \leq C(1 - z_i) \ \forall \ i = 1, \ldots, M, \ l = 1, \ldots, M_l$$

$$|u_i| \leq u_{\text{max}} \ \forall \ i = 0, \ldots, N - 1$$

$$z_i \in \{0, 1\} \ \forall \ i = 1, \ldots, M$$

Problem 2 has a large number of variables that grow as the number of particles $M$ and the number of time steps $N$
increase, and is therefore still limited in terms of how many particles can be used and the amount of time given to reach the target. For instance, using 500 particles over \( N = 5 \) time steps, there are a total of 510 variables: 500 indicator variables \( z_i \), one for each particle, and 10 variables for \( U \), since \( U \in \mathbb{R}^{10} \). Further, to define inequality constraints ensuring \( X_1, \ldots, X_4 \in K \) and \( X_5 \in T \), we need \( M_l = 48 \) \((a_l, b_l)\) pairs of inequalities, producing 500 \( \times \) 48 = 24000 constraints of the form \( a_l^T X(i) - b_l \leq C(1 - z_i) \). However, using commercially available MILP solvers, such as CPLEX [17], for a single given \( X_0 \) this problem was solved in 6.88 seconds on a 2 GHz Macbook with 2 GB of RAM.

One advantage to the particle method is that accuracy can be traded for computation time. Fewer particles can lead to a quick approximation of the reach-avoid probability for different \( X_0 \), and those \( X_0 \) that seem to produce larger probabilities can be recalculated with more particles to obtain a more accurate result if desired.

**B. Convex Chance-Constrained Approximation**

An alternative to the particle approximation approach taken above is to reformulate Problem 1 by moving \( p_{X_0}^N(K, T) \) from the objective function to a chance constraint that must be enforced with probability \( 1 - \alpha \).

**Problem 3 (Chance Constrained Formulation of Problem 1)**

\[
\begin{align*}
\min & \quad \alpha \\
\text{Subject to:} & \quad p_{X_0}^N(K, T) \geq 1 - \alpha \\
& \quad |u_i| \leq u_{\max} \quad \forall i = 0, \ldots, N - 1 
\end{align*}
\]  

(13)

Generally, this would not make the problem easier to solve, since we must still evaluate \( p_{X_0}^N(K, T) \). However, because of the linearity and Gaussian noise, we can approximately solve Problem 1 by breaking (13) into univariate Gaussian constraints, which are convex (see [12], [13]). To see this, use the converse of (12) and Boole’s inequality to write

\[
\begin{align*}
X_{1:N-1} & \notin K \vee X_N \notin T \Rightarrow X \in \bigcup_l \{ X : a_l^T X > b_l \} \\
P[X_{1:N-1} \notin K \vee X_N \notin T] & = P[l \bigcup \{ X : a_l^T X > b_l \} ] \\
& \leq \sum_{l=1}^{M_l} P[a_l^T X > b_l] 
\end{align*}
\]  

(14)

The inequality in the above expression indicates an upper bound on the probability of not being in \( K \) or \( T \) at time \( N \), and therefore a lower bound on the actual reach-avoid probability, which is still desirable.

Noting that \( a_l^T X \) is a scalar, and \( a_l^T X = a_l^T (X_0 + HU + GV) \), it follows that \( a_l^T X \sim \mathcal{N}(a_l^T X_0 + HU, a_l^T GSG^T a_l) \) has a univariate Gaussian distribution. Applying (14), constraint (13) can be rewritten as follows.

\[
\begin{align*}
p_{X_0}(K, T) & \geq 1 - \sum_{l=1}^{M_l} P[a_l^T X > b_l] \\
& \geq 1 - \alpha
\end{align*}
\]

By taking a “risk allocation” approach as in [11], we can allow for a different probability of violating each individual constraint, i.e. for each \( l \), \( P[a_l^T X > b_l] \leq \alpha_l \), with \( \sum \alpha_l = \alpha \). We then require each \( \alpha_l \geq 0 \) and \( \sum \alpha_l = \alpha \leq 1 \), leading to the following approximation to Problem 3.

**Problem 4 (Chance Constrained Convex Approximation to Problem 1)**

\[
\begin{align*}
\min & \quad \sum_{l=1}^{M_l} \alpha_l \\
\text{Subject to:} & \quad 1 - \Phi \left( \frac{b_l - a_l^T X}{a_l^T GSG^T a_l} \right) \leq \alpha_l \quad \forall l = 1, \ldots, M_l \\
& \quad |u_i| \leq u_{\max} \quad \forall i = 0, \ldots, N - 1 \\
& \quad \sum_{l=1}^{M_l} \alpha_l \leq 1 \\
& \quad \alpha_l \geq 0 \quad \forall l = 1, \ldots, M_l 
\end{align*}
\]

The function \( \Phi(\cdot) \) indicates the standard normal cumulative distribution function, which is a concave function so long as its argument is non-negative (\( \Phi(x) \) is concave for all \( x \geq 0 \)), implying that \( 1 - \Phi(x) \) is convex for all \( x \geq 0 \). Problem 4 is therefore convex so long as \( b_l - a_l^T X \geq 0 \), or \( \alpha_l \leq 0.5 \) for all \( l \). If any \( \alpha_l \) is greater than 0.5, the reach-avoid probability for the given \( X_0 \) will be quite low, so that we will likely only be concerned with those instances of Problem 4 when the problem is indeed convex, and hence we are guaranteed a solution.

Compared to the particle approach, Problem 4 has far fewer variables and constraints. For the same case with \( N = 5 \) time steps, we again have \( M_l = 48 \) constraints \( a_l^T X \leq b_l \) confining \( X_1, \ldots, X_4 \in K \) and \( X_5 \in T \). Associated to each constraint is a variable \( \alpha_l \), so that there are \( 48 + 10 = 58 \) variables total (and again the 10 variables correspond to \( U \in \mathbb{R}^{10} \)). The drawback here is the nonlinear constraint involving the Gaussian normal density function, \( 1 - \Phi \left( \frac{b_l - a_l^T X}{a_l^T GSG^T a_l} \right) \leq \alpha_l \). Nonlinear solvers can be more time consuming if the initial solution point provided to the algorithm is far from the optimal solution. For an initial solution generated randomly, and the same \( X_0 \) as was implemented in the particle method, Problem 4 was solved in 2.81 seconds using MATLAB’s \texttt{fmincon} function and its active-set algorithm. However, for some initial points the algorithm can fail to converge completely and the process must be repeated with a different initial point, adding to the overall time of the algorithm. With the same \( X_0 \) but a different initial point, the problem was solved in 50.99 seconds with 7 different initial points before finding the solution. Using different algorithms or software and “good” initial guesses may help speed up the algorithm.
C. Particle Approximation Using Feedback

We now show how to calculate (5) when \( U = KX + U_0 \) with \( K \) block lower triangular, and \( u_k \) bounded between known maximum and minimum values, i.e. \( U = \{K, U_0 : U_{\text{min}} \leq KX + U_0 \leq U_{\text{max}}\} \). We only consider the particle approximation method in the case of feedback, because the chance-constrained method will no longer be convex, and the constraint in Problem 4 involving \( \Phi(\sum_{i=1}^{a_k} X^{(i)} + u_k) \) becomes difficult to enforce. Problem 2, however, can be modified nicely to accommodate a feedback controller.

First, the expression for \( X \) must be modified to incorporate feedback in (9). \( X \) and \( U \) can be written as affine functions of the random vector \( V \), as shown in [16]:

\[
\begin{align*}
[X] & = PV + \begin{bmatrix} \bar{X} \\ \bar{U} \end{bmatrix} \\
\bar{P} & = \begin{bmatrix} G + HK(I - HK)^{-1}G \\ K(I - HK)^{-1}G \end{bmatrix} \\
\bar{X} & = X_0 + HU_0 + HK(I - HK)^{-1}(X_0 + HU_0) \\
\bar{U} & = K(I - HK)^{-1}(X_0 + HU_0) + U_0
\end{align*}
\]

Since these equations are not convex in the variables \( K \) and \( U_0 \), new variables \( Q = K(I - HK)^{-1} \) and \( R = (I + HQ)U_0 \) are introduced to make the equations convex. The variables of interest, \( K \) and \( U_0 \), can be recovered after solving for optimal \( Q \) and \( R \), as in [16]. Note that \( U \) is now random, because it is a function of the Gaussian vector \( V \). We therefore cannot impose with certainty constraints on the maximum and minimum values of \( U \). We instead require \( E[\|U\|] \leq U_{\text{max}} \), which can also be evaluated in the manner of (10). Problem 2 is reformulated as:

Problem 5 (Particle Approximation with Feedback)

\[
\max \sum_{i=1}^{M} z_i
\]

Subject to:

\[
\begin{align*}
X^{(i)} & = (I + HQ)GV^{(i)} + (I + HQ)X_0 + HR \\
U^{(i)} & = QGV^{(i)} + QX_0 + R \\
\alpha_i^T X^{(i)} - b_i & \leq C(1 - z_i) \\
& \forall i = 1, \ldots, M, \\
& l = 1, \ldots, M_l
\end{align*}
\]

\[
\frac{1}{M} \sum_{i=1}^{M} |U^{(i)}| \leq U_{\text{max}}
\]

\[
z_i \in \{0, 1\} \quad \forall i = 1, \ldots, M
\]

\( Q \) block lower triangular

The number of variables and constraints significantly increases in comparison to Problem 2, as does the solution time. For the same initial state \( X_0 \) and number of particles as the open loop particle method, Problem 5 was solved in 80.04 seconds.

IV. RESULTS

Both of the open loop methods described above are applied to the reachability problem for the final phase of in-plane rendezvous, as described in [1]. First, we define the safe set \( K \) that the deputy should remain within for all time steps 0 to \( N - 1 \), and the target set \( T \) next to the chief, that should be reached at time \( N \). The set \( K \) is an LoS cone at the origin in \( x \) and \( y \) (independent of time), and a square in \( \dot{x} \) and \( \dot{y} \), i.e. the maximum and minimum velocities are bounded for all time steps. The target set \( T \) is a four-dimensional rectangle adjacent to the origin. We set \( \Sigma = \text{diag}[1e^{-4}, 1e^{-4}, 5e^{-8}, 5e^{-8}] \). The noise covariance is kept small, although it could easily be altered, since open-loop controllers do not deal well with noise over longer periods of time (and the covariance grows with \( N \)). As the covariance grows, the potential distance between different realizations of the trajectory \( X \) grows as well, and it is harder for one controller, without feedback, to drive all possible realizations to the origin while keeping them within an LoS cone.

Problem 2 and Problem 4 are set up accordingly, and solved over a mesh of initial \( X_0 \) values ranging from 2 km behind the chief to directly behind it. We compute the reachable sets over a period of \( N = 5 \) time steps, \( \Delta = 20 \) seconds per time step (as in [1]), and compare the particle approach to the convex optimization. Problem 2 is solved in CPLEX and Problem 4 using an active-set algorithm through \texttt{fmincon} in MATLAB’s Optimization toolbox. We started with a coarse grid for \( X_0 \), with \( x_0 \) equally spaced 0.1 km apart ranging from \(-2 \) to \( 2 \) km, \( y_0 \) ranging from \(-2 \) km to \( 0 \) km with the same spacing as \( x_0 \), and \( \dot{x}_0 \), \( \dot{y}_0 \) ranging from \(-0.2 \) km/s to \( 0.2 \) km/s, equally spaced 0.01 km/s apart. We isolated the states \( X_0 \) leading to higher reach-avoid probabilities, refined the mesh limited to these values, and recalculated the reach-avoid probabilities.

Fig. 1 shows the probability of staying within \( K \) and reaching \( T \) in 5 time steps, over varying \( x_0 \) and \( y_0 \) (using the refined mesh with 0.01 km between points), with fixed \( \dot{x}_0 = \dot{y}_0 = 0.01 \) km/s, using the particle approach with 800

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Fig. 2. The sets $RA_{0.3}^{5}(K,T)$ and $RA_{0.2}^{5}(K,T)$ (i.e. the sets of all $X_0$ such that $p_{X_0}(K,T) \geq 0.7$ and $p_{X_0}^{5}(K,T) \geq 0.8$) with $\dot{x}_0 = 0.01$ km/s, $\dot{y}_0 = 0.01$ km/s in (a), and $\dot{x}_0 = -0.01$ km/s, $\dot{y}_0 = 0.01$ km/s in (b), fixed. The reach-avoid sets for $\alpha = 0.3$ and $\alpha = 0.2$ are shown for the convex approximation (black, blue), and for the particle approximation using 800 particles (red, green) methods. In both figures, the $RA$ sets span only a few m/s in each direction, demonstrating the importance of accuracy in the initial velocities for rendezvous to be successful.

Fig. 3. The sets $RA_{0.3}^{5}(K,T)$ and $RA_{0.2}^{5}(K,T)$ (i.e. the sets of all $X_0$ such that $p_{X_0}(K,T) \geq 0.7$ and $p_{X_0}^{5}(K,T) \geq 0.8$) with $x_0 = -0.9$ km, $y_0 = -1$ km in (a), and $x_0 = 0.9$ km, $y_0 = -1$ km in (b), fixed. The reach-avoid sets for $\alpha = 0.3$ and $\alpha = 0.2$ are shown for the convex approximation (black, blue), and for the particle approximation using 800 particles (red, green) methods. In both figures, the $RA$ sets span only a few m/s in each direction, demonstrating the importance of accuracy in the initial velocities for rendezvous to be successful.

particles. The area of positive probability is limited to where both $x_0$ and $y_0$ are negative, because the initial velocities are both positive, and over the time period of $N = 5$, the controller cannot sufficiently reverse the velocity of the deputy to enable it to still reach the origin. Further, the $x_0$ and $y_0$ leading to positive probability only lie in a range of approximately $-1.1$ km to $-0.8$ km, somewhat in the middle of the LoS cone. This is because we have restricted the deputy to rendezvous with the chief at exactly $N = 5$ time steps. Starting too far away, the controller cannot bring the deputy to the chief within the small time frame, and starting too close, the controller cannot prevent the deputy from overshooting the chief, hence the small region with positive probability of success. Taking the union over $N$ of all reach-avoid sets with $N \leq 5$, as mentioned in Section II-B, would expand the full reach-avoid set of Fig. 1. Finally, there is no initial position leading to a reach-avoid probability higher than 0.9, even with the choice of a small noise covariance. Without feedback, the reach-avoid probability will grow smaller as $N$ increases, which is already evident in just over 5 time-steps.

Fig. 2a shows specific reach-avoid probabilities (for a given probability $1 - \alpha$), computed with the particle and convex approaches. The set of all initial states $x_0$, $y_0$ leading to trajectories that will stay within $K$ and reach $T$ at $N = 5$, with probability 0.7 and 0.8 when $\dot{x}_0 = \dot{y}_0 = 0.01$ km/s, is shown. The sets generated by the particle approach (using 800 particles) are slightly larger than those from the convex approach, consistent with the fact that the convex approach gives a slight under-approximation of the reach-avoid set. A comparable scenario, with $\dot{x}_0 = -0.01$ km/s, is shown in Fig. 2b. Note in this case that the reach-avoid sets occur in the region where $x_0$ is positive, for the same reason that they appear where $x_0$ is negative when the initial $x$-velocity is positive.

The reach-avoid sets are not symmetric, because the zero-input dynamics also are not symmetric. Plotting sample trajectories of the zero-input, zero-noise CWH equations demonstrates this. Fig. 4 shows $RA_{0.2}^{5}(K,T)$ for $\dot{y}_0 = 0.01$ km/s, computed with the particle approach. As $\dot{x}_0$ decreases,
the size of the reach-avoid set shrinks in the $y$-direction, as in Fig. 2. The scale of the axis shows that the range of $\dot{x}_0$ for a given $x_0$ and $y_0$ is much smaller relative to the ranges of $x_0$ and $y_0$ when $x_0$ is fixed. This indicates that the reach-avoid probability is much more sensitive to the initial velocity than it is to the initial position. Fig. 3 explores this further.

Reach-avoid sets for fixed $x_0$, $y_0$ and varying $\dot{x}_0$, $\dot{y}_0$ using a mesh with 1 m/s between points using both the convex and particle approaches are given in Fig. 3. In Fig. 3a, $x_0 = -0.9$ km and $y_0 = -1$ km, and in Fig. 3b, $x_0$ is +0.9 km. The reach-avoid sets using the particle approach are approximately symmetric across the two values of $x_0$. The convex reach-avoid sets in Fig. 3a are slightly misshapen relative to all the other reach-avoid sets, which is unfortunately caused by the MATLAB algorithm, which is unable to find an optimal solution to Problem 4 for various initial guesses. Regardless, it is clear that for a given starting position $x_0$, $y_0$, the initial velocities must be specified to within an accuracy of a couple m/s, or the probability of success drops drastically, and in fact will drop to zero very quickly in the case of an open loop controller.

To compare the performance of the open loop controller to one that uses feedback, we next present results for the reach-avoid set calculated by solving Problem 5, in Fig. 5. We fix $\dot{x}_0 = \dot{y}_0 = 0.01$ km/s in Fig. 5a, and $x_0 = -0.9$ km, $y_0 = -1$ km in Fig. 5b. All parameters (i.e. $\Sigma$ and $N$) are kept the same as in the open loop case, although the number of particles is reduced to 500, and a slightly coarser grid is used, in order to decrease computation time. A feedback controller drastically increases the size of the reach-avoid set, because it can steer those trajectories driven away from the origin by the process noise back to the desired path. This contrast in results between controllers highlights the importance of considering the type and effect of the controller to avoid overly conservative reachability results, and merits further discussion. It is therefore an area of potential future work.

Finally, we briefly compare the performance of both methods used to produce the above results in the open loop case. Both methods produce approximations to the true reach-avoid probabilities. Figs. 2 and 3 show that the convex chance-constrained approach consistently overapproximates (as expected) the reach-avoid probabilities relative to the particle approximations, although never to the point of being unreasonable. We found that the particle method was consistent in its results, and has the advantage of allowing for a quick, coarse approximation using less particles, to then narrow down the region where the reach-avoid probabilities should be calculated more accurately. Both methods have the disadvantage that, if the reach-avoid probability for a given initial point is close to zero, they will possibly never converge to a solution, particularly in the case of Problem 4, since the problem is in fact no longer convex. Hence we found the need to stop the algorithm after it had taken sufficient time to find an optimal solution, were one to exist. The convex method was generally faster, but only when the initial values over which the reach-avoid sets were calculated had been narrowed down enough to eliminate most of the low probability calculations, and when “good enough” initial solutions were provided to avoid recalculating the reach-avoid probability several times. The data for Figs. 2 and 3 generally took about 20 minutes each to calculate (with no significant difference in time between the methods). However, the particle approach was used to generate Fig. 1 and Fig. 4 because over a wide range of values it was found to be faster.

While the results presented are for a four-dimensional system, both approaches should be suitable for reachability calculations of higher dimensional systems as well. The number of constraints in either Problem 2 or Problem 4 will not increase for higher dimensional systems (except possibly in the number of control inputs $u$ to be bounded, which is insignificant compared to the other constraints). The main computation will lie in calculating the actual reach-avoid set, where the problem must be solved repeatedly over a much larger grid of $X_0$ values. Again, the process can be sped up by solving over a coarse grid, and in the particle approximation case using a small number of particles, and then focusing and refining the grid to where the success probabilities are estimated to be highest.

Further, both methods can be used for reachability calculations of any linear system with additive Gaussian noise. The convex chance-constrained approach of Problem 4 will only work with Gaussian noise, but the particle method could be applied to noise following any known distribution. We required the reach and the complement of the avoid set to be convex, but nonconvex regions could also be addressed by decomposition into multiple convex regions (see, e.g. [18]).

V. CONCLUSIONS

Two novel methods for reachability calculations on higher-dimensional linear stochastic systems have been presented. One involves sampling from the noise distribution to generate a particle approximation to the expected value (6), which

![Fig. 4](image-url) Three-dimensional reach-avoid set $RA_0^2(K, T)$, with $y_0$ fixed at 10 m/s, generated using the particle approximation method. As $x_0$ ranges from negative to positive, $\dot{x}_0$ ranges from positive to negative. The cross-sections for fixed $\dot{x}_0$ shrink as $x_0$ becomes positive due to the asymmetric nature of the zero-input dynamics. The difference in cross-sections for $x_0$ positive versus negative are highlighted in Fig. 2.
would otherwise be too difficult to evaluate, and using mixed integer linear programming to solve the optimization problem (5). The other exploits the linearity of the dynamics and the Gaussian assumption on the noise to produce a convex optimization problem also approximating (5), by making the reach-avoid probability a chance constraint whose bound is actually the objective to be maximized. An open-loop control vector $U^*$ was found to maximize the reach-avoid probability in both cases.

The obvious setback is that with the propagation of the noise, an open-loop controller cannot optimally control all realizations of the dynamics over an extended length of time. The noise leads to dynamics which diverge over time, so that the chances of reaching the target set $T$ using an open loop controller go to zero very quickly as the time horizon is extended.

We were able to impose a state-based feedback controller using the particle approximation method, despite the increasing number of variables and constraints. The natural next step is then to look more closely at closed-loop controllers. While the particle approximation method still works, the convex approximation technique breaks down when feedback is introduced, because the constraints are no longer convex. One possible approach to maintain convexity would be to use a model predictive controller to simulate feedback (although it may not help the speed of the algorithm), to see how this improves the stochastic reach-avoid probability.

REFERENCES


