Tracking Error Analysis for Singularly Perturbed Systems Preceded by Piecewise Linear Hysteresis

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Abstract—In this paper we introduce a new method for analyzing the closed-loop control system that involves a singularly perturbed plant preceded by hysteresis nonlinearity with piecewise linear characteristics, an example of which is a piezo-actuated nanopositioners. Different methods are compared to quantify the tracking error and examine how the change in slopes from one segment to another interacts with the controller parameters and hence affects the tracking error. These methods all involve the combination of inverse hysteresis compensation and feedback control, but the points of insertion for the hysteresis inverse vary. A proportional-integral feedback controller is used throughout this comparison. The presented analysis is important because it provides an explicit expression for the tracking error, where the feedback controller parameters can be adjusted for the desired performance. Simulation and experimental results are presented for tracking control of a piezo-actuated nanopositioner, where the hysteresis is modeled with a Prandtl-Ishlinskii (PI) operator. The analysis carried out in this paper is applicable to other operators, such as the modified PI-operator and the Krasnoselskii-Porkovskii (KP) operator among others, since they all demonstrate piecewise linear hysteresis characteristics.

I. INTRODUCTION

Piezoelectric actuators are commonly used in nanopositioning applications such as scanning tunneling microscopes and atomic force microscopes. They have large bandwidth and can produce large mechanical forces [1], [2]. However, they exhibit non-desirable behaviors such as hysteresis, creep, and vibrations. Hysteresis nonlinearity heavily affects the positioning precision. Therefore, in the last two decades, research has been widely conducted for both hysteresis modeling and control design.

Control methods used for mitigating the hysteresis effect are classified in general under three categories [1], which are feedback control, feedforward compensation, and the integration of both. Figure 1 illustrates the feedforward scheme augmenting feedback control. The feedback control can be as simple as integral control [3] or more sophisticated, such as adaptive control [4]–[7], servo-compensator [8], and sliding mode control (SMC) [9]–[13].

In this paper, we conduct the analysis of the tracking error for several control schemes that combine the hysteresis inversion with feedback control. We consider a plant that consists of linear dynamics preceded by hysteresis nonlinearity, as has often been proposed for smart material-actuated systems [1], [7], [14]–[16]. In the first scheme (Scheme 1), the hysteresis inversion is placed inside the feedback loop, following the feedback controller and preceding the hysteresis nonlinearity. In the second scheme (Scheme 2), the hysteresis inversion is placed in the feedforward path. Both Scheme 1 and Scheme 2 have been used extensively in the literature with demonstrated success in experiments [17]–[20]; however, rigorous analysis on these schemes has been limited. We also suggest a third scheme (Scheme 3), where hysteresis inversion is placed in both the feedforward path and the feedback path.

Motivated by the properties of piezo-actuated nanopositioning systems, we assume that the linear dynamics of the plant are stable and have large bandwidth. This assumption allows us to use singular perturbation analysis to obtain low-frequency approximation of the solution in which the linear dynamics are neglected. For higher frequency references we can use singular perturbations to improve the accuracy of the approximation.

In addition, we assume that the hysteresis nonlinearity has piecewise linear characteristics; in other words, all hysteresis loops (major loops or minor loops) consist of linear segments, where each segment $s_i$ has a slope $m_i$ and an intercept $\gamma_i$ with the output axis. See Fig. 2 for illustration. This assumption is used in [21] where an adaptive inverse control scheme is presented to identify the hysteresis slopes and intercepts in order to compute the hysteresis inverse. The assumption also holds true for a wide class of hysteresis operators, such as the Prandtl-Ishlinskii (PI) operator [19], the modified PI operator [22], and the Krasnoselskii-Porkovskii (KP) operator among others [14]. In our analysis, we further assume that, when the hysteresis nonlinearity is known, its inverse can be constructed exactly. This assumption holds true for many operators, examples of which include the PI operator [19] and modified PI operator [22].

In our analysis, we consider both the case where the
current hysteresis segment is precisely known and the case
where the segment is not known exactly. Simulation and
experimental results on commercially available nanoposition-
er demonstrate that the proposed new scheme (Scheme 3)
delivers better performance compared with the other two
schemes and other methods based on servo-compensation
and sliding-mode control.

The remainder of the paper is organized as follows. In
Section II, we briefly describe the system setup. This is
followed by the analysis of the three methods in Sections III,
IV, and V. Simulation and experimental results are presented
in Section VI. Finally, we provide concluding remarks in
Section VII.

II. SYSTEM MODEL

The linear dynamics of the plant are represented by a
singularly perturbed linear system.

\[
\begin{align*}
\epsilon \dot{z} &= Az + Bu, \\
y &= Cz
\end{align*}
\]

(1)

where \(A\) is a Hurwitz matrix, \(B\) and \(C\) are matrices with
proper dimensions, and \(z\) is the state vector. We assume
that the feedback controller is a proportional-integral controller,
represented as

\[
\begin{align*}
\dot{x} &= e = y_r - y \\
w &= k_ix + k_pe
\end{align*}
\]

(2)

where \(e\) is the tracking error and \(w\) is the PI-controller output.
A hysteresis operator \(\Gamma_p\), with piecewise linear characteris-
tics, precedes the linear dynamics. We denote the identified
hysteresis operator as \(\Gamma_m\) and its inverse as \(\Gamma^{-1}_m\). Of course,
when we know the hysteresis nonlinearity fully, \(\Gamma_m = \Gamma_p\);
otherwise, there is a mismatch representing the model uncer-
tainty. The input-output relationship of \(\Gamma_m\) can be described
in each segment of a hysteresis loop as follows

\[
u = m_i y + \gamma_i
\]

(3)

for some \(m_i\) and \(\gamma_i\). We assume \(m_i \neq 0\). For convenience,
we will drop the subscript \(i\) in the analysis, where \(m\) and

\(\gamma\) denote the slope and intercept of the line segment under
consideration. Hence, the following analysis will be valid for
any segment.

By letting \(\epsilon \rightarrow 0\) in Eq. (1) we obtain the DC gain (\(h\)) of the
plant as

\[
h = -CA^{-1}B
\]

(4)

III. SCHEME 1: HYSTERESIS INVERSE IN THE FEEDBACK PATH

Fig. 3 describes the system configuration for Scheme 1,
where \(\frac{1}{\epsilon}\) is used to compensate the DC gain of the linear
dynamics. The input to the inverse operator, \(u_d\), and its
output, \(v\), are expressed as

\[
v = \frac{1}{m}(u_d - \gamma)
\]

(5)

\[
u_d = \frac{1}{\epsilon}v_r + k_ix + k_pe
\]

(6)

Let us denote the corresponding slope of the perturbed
model \(\Gamma_p\) by \(\tilde{m}\) and the intercept by \(\tilde{\gamma}\), where \(\tilde{m} = m + \Delta_m\)
and \(\tilde{\gamma} = \gamma + \Delta\gamma\), then

\[
u = (m + \Delta_m)v + (\gamma + \Delta\gamma)
\]

(7)

By substituting \(u_d\) from (6) and \(v\) from (5) into (7), we express \(u\) as

\[
u = \frac{m + \Delta_m}{m} \left[ \frac{1}{h} y_r + k_ix + k_pe(y_r - Cz) \right] + \frac{m\Delta\gamma - \gamma\Delta_m}{m}
\]

(8)

The singularly perturbed closed-loop system is obtained by
inserting \(u\) from (8) into (1)

\[
\begin{align*}
\dot{x} &= y - Cz \\
\epsilon \dot{z} &= \left[ A - \frac{k_p(m + \Delta_m)}{m} BC \right] z + B \frac{m + \Delta_m}{mh} (1 + k_ph)y_r \\
&+ B \frac{m + \Delta_m}{m} k_ix + B \frac{m\Delta\gamma - \gamma\Delta_m}{m}
\end{align*}
\]

(9)

For the fast model of the singularly perturbed system to
be exponentially stable, we assume that the matrix \((A -
\( kp(m+\Delta m)BC \) is Hurwitz. To obtain the slow model we set \( \varepsilon = 0 \), to get
\[
\begin{align*}
    z &= -\left[ A - \frac{k_p(m+\Delta m)BC}{m} \right]^{-1} B \left[ \frac{m+\Delta m}{m} (1+k_p h) y_r + k_p B \frac{(m\Delta y - \gamma\Delta m)}{m} \right] - k_p x + B \left( \frac{m\Delta y - \gamma\Delta m}{m} \right) \\
    &= B \left( \frac{m\Delta y - \gamma\Delta m}{m} \right)
\end{align*}
\] (10)

We insert the fast variable \( z \) from (10) into the \( \dot{x} \) equation (9) and obtain the slow (low frequency) model as
\[
\dot{x} = -\frac{(m+\Delta m)k_i}{m+k_p(m+\Delta m)h} x + \frac{-\Delta m}{m(1+k_p h)} y_r
\]
+ \( h \frac{m\Delta y - \gamma\Delta m}{m+k_p(m+\Delta m)h} \) (11)

By letting \( \Delta_m = 0 \) and \( \Delta_y = 0 \) in (11), we can obtain \( \dot{x} \) for the case of a non-perturbed operator as
\[
\dot{x} = -\frac{k_i}{1+k_p h} x
\] (12)

Using the solution of (12), the slow variable \( x \) is given by
\[
x(t) = x(0) e^{-\frac{k_i}{1+k_p h} t} + O(\varepsilon)
\] (13)

We see from (11) (with \( \Delta_m = 0 = \Delta_y \)) and (13) that neither the tracking error nor its integral is a function of the reference \( y_r \), or the intercepts \( \gamma \) of the hysteresis segments. It has only a decaying term dependent on the initial value \( x \) but independent of the segment’s slope \( m \).

The linear dynamics which are ignored by the singular perturbed solution is abstracted in the term \( O(\varepsilon) \). It is important to consider this term at high frequencies. All tracking error terms caused by the hysteresis nonlinearity and any other external disturbances, which are usually small compared with the hysteresis nonlinearity, can be compensated for by the gains \( k_i \) and \( k_p \). However, \( k_p \) might be constrained by the stability of the system.

Turning now to the case of a perturbed operator, we find that (11) has two extra terms compared with (12). The second term shows that the tracking error \( e = \dot{x} \) will be a function of the reference signal. However, the integral action can compensate for the third term at the steady state since it is a constant.

IV. SCHEME 2: HYSTERESIS INVERSE IN THE FEEDFORWARD PATH

In this method the inverse operator is inserted in the feedforward path as shown in Fig. 4. Since the signal applied to the operator consists of both the hysteresis inverse output and the PI-controller output, there is mismatch between the internal states of \( \Gamma_m \) and \( \Gamma_p \), which prevents perfect inversion.

We will follow the same steps as in Section III to determine the tracking error. From Fig. 4, \( v \) can be expressed as
\[
v = \frac{1}{m} \left( \frac{1}{1+k_p h} y_r - \gamma \right) + k_i x + k_p e
\] (14)

By substituting \( v \) from (14) into (7), we get \( u \) as
\[
u = (m+\Delta_m) \left[ \frac{1}{m} \left( \frac{1}{1+k_p h} y_r - \gamma \right) + k_i x + k_p e \right] + (\gamma + \Delta_y)
\]
\[
u = \frac{m+\Delta_m}{m} \left[ \frac{1}{h} + k_p (m+\Delta_m) \right] y_r
\]
+ \( k_i (m+\Delta_m) x + k_p (m+\Delta_m) C z + \frac{m\Delta y - \gamma\Delta m}{m} \) (15)

The closed-loop system of the perturbed system is obtained by inserting \( u \) from (15) into (1)
\[
\dot{x} = -\gamma C z
\]
\[
\varepsilon \dot{z} = [A - kp(m+\Delta m)BC] z + B \left[ \frac{(m+\Delta_m)(\frac{1}{h} + k_p h)}{m} y_r \right.
\]
+ \( (m+\Delta_m k_i) x + \left( \frac{m\Delta y - \gamma\Delta m}{m} \right) \) (16)

We assume that \( (A - kp(m+\Delta m)BC) \) is Hurwitz and set \( \varepsilon = 0 \).

\[
z = -\left[ A - kp(m+\Delta m)BC \right]^{-1} \left[ \frac{(m+\Delta_m)(\frac{1}{h} + k_p h)}{m} y_r \right.
\]
+ \( (m+\Delta_m k_i) x + \left( \frac{m\Delta y - \gamma\Delta m}{m} \right) \) (17)

We insert the fast variable \( z \) from (17) into \( \dot{x} \)-equation of (16) and obtain the slow model:
\[
\dot{x} = \left[ A - kp(m+\Delta m)BC \right]^{-1} \left[ \frac{(m+\Delta_m)(\frac{1}{h} + k_p h)}{m} y_r \right.
\]
+ \( k_i (m+\Delta_m) x + \left( \frac{m\Delta y - \gamma\Delta m}{m} \right) \) (18)

By letting \( \Delta_m = 0 \) and \( \Delta_y = 0 \), Eq. (18) reduces to
\[
\dot{x} = -\frac{k_i h}{1+k_p h} x
\] (19)

implying
\[
x(t) = x(0)e^{-\frac{k_i h}{1+k_p h} t} + O(\varepsilon)
\] (20)

Comparing (13) and (20), we see that in Scheme 2 the tracking error is affected by the slope of each segment.
even when the operator is unperturbed. This leads to the conclusion that Scheme 1 is better than the Scheme 2, and simulation results later in this paper will confirm this observation.

We can further compare the two schemes when the hysteresis operator is perturbed. Equations (18) and (11) are equivalent when the slope of the segment, \( m = 1 \), for the same amount of perturbation except, for the constant term, which has opposite signs. Both equations show that the tracking error is proportional to the reference signal \( y_r \), and the coefficient of this term becomes larger as \( m \) becomes smaller. The error contribution from the latter term achieves the maximum when \( |y_r| \) reaches its maximum. Equation (18) shows that Scheme 2 has larger coefficient associated with the \( y_r \) term, and hence it would have larger tracking error, which is confirmed by our later simulation results.

V. SCHEME 3: HYSTERESIS INVERSION IN BOTH FEEDBACK AND FEEDFORWARD PATHS

Fig. 5 illustrates our proposed idea of placing hysteresis inversion in both the feedback and feedforward paths. Let us denote the slope of inverse-operator in the feedback branch by \( m_l \) and the intercept by \( \gamma_l \), and analyze the system with a perturbed operator. First note that

\[
v = \frac{1}{m} (y_r - \gamma) + \frac{1}{m_l} (k_i x + k_p e - \gamma_l)
\]

By inserting (21) into the operator equation (3), we obtain

\[
u = m \left[ \frac{1}{m} (y_r - \gamma) + \frac{1}{m_l} (k_i x + k_p e - \gamma_l) + \gamma_l \right]
\]

By following similar steps as in the previous sections, we arrive at the slow model

\[
\dot{x} = \frac{1}{k_p (m + \Delta m) h} \left[ \frac{1}{m_l h} k_i x - \frac{-\Delta_m}{m (1 + k_p (m + \Delta m) h)} y_r \right.

+ \frac{h}{m (1 + k_p (m + \Delta m) h)} y + h \frac{(m + \Delta m) y}{m (1 + k_p (m + \Delta m) h)} + h \frac{(m + \Delta m) \gamma}{m (1 + k_p (m + \Delta m) h)}
\]

If \( m_l < m \) in Eq. (23), then third scheme will have less tracking error than the others because the coefficient of the driving term becomes smaller. Although simulations show that the tracking error is smaller for all segments, further analysis of the operator should be conducted in order to find the relationship between the slopes \( m_l \) and \( m \) for each segment.

VI. SIMULATION AND EXPERIMENTAL RESULTS

The simulation is based on the model and parameters identified experimentally for a commercial nanopositioner (Nano-OP65 with Nano Drive controller, Mad City Labs Inc.) The linear dynamics are fitted with forth-order singular perturbed system with \( \varepsilon = 7.63 \times 10^{-5} \). The PI hysteresis is modeled with 10 play operators that has the threshold vector \( r = [0, 0.63, 1.27, 1.9, 2.54, 3.18, 3.81, 4.45, 5.09, 5.73]^T \) and the vector of weights for the operator is \( w^T = [5.88, 1.58, 0.47, 0, 0.98, 0, 0, 0.4, 0, 0] \).

A. Simulation Results

A sinusoidal reference signal with amplitude of 50 \( \mu \)m and frequency of 100 Hz is applied to the system. Parameters for this simulation are chosen as \( k_i = 100 \) and \( k_p = 2 \). In Fig. 6 we compare the three schemes when the operator is not perturbed. Scheme 1 shows slightly better tracking performance than Scheme 2. This is because the slopes are close to one. We can see that for Scheme 1, the error appears
as a sinusoidal signal and is not affected by the slope of each segment as predicted in (12). Scheme 3 shows better performance than 1.

Fig. 7 shows similar comparison but with the perturbed operator. There are two things we can observe here. First, the difference in the tracking error between schemes 1 and 2 is still small for the same reason that the slopes are close to one. Second, the size of the error for each segment becomes dependent on the segment slope and may be large or small depending on the value of $m$ at that segment.

B. Experimental Results

In experiments sinusoidal signals are used as reference trajectories in order to compare the proposed method with other methods that have been applied to the same nanopositioner. Fig. 8 and Fig. 9 show the results for the cases of tracking 5 Hz and 100 Hz signals, respectively. The maximum tracking error is 0.632% for 5 Hz while it is 2.06% for 100 Hz. We notice that the measurement noise has high spikes and affects the overall value of the maximum tracking error at low frequencies. Still, this has small effect when we calculate the mean error of the tracking error.

In Tables I and II, we list more experimental results for the proposed method (Scheme 3) and compare them with servo-compensator methods presented in [8] and Sliding-Mode-Control (SMC) presented in [23] that are implemented in the same nanopositioner used in this work. There are two designs for the servo-compensator: Single Harmonic Servo-Compensator (SHSC) and Multi-Harmonic Servo-Compensator (MHSC). It can be seen that the Scheme 3 delivers better results than SMC and SHSC for all frequencies, and better results than MHSC at relatively low frequencies (5 Hz and 25 Hz), while the tracking performance of MHSC is in general better at higher frequencies. We should note that while the servo-compensators are designed for periodic references of given frequencies, there is no such limitation for SMC or Scheme 3.

VII. Conclusion

Analysis of tracking error is important for quantifying the effect of hysteresis nonlinearity on the overall tracking performance of piezoactuator. In this paper we analyzed several control schemes that augment feedback control with inverse hysteresis compensation. In all of these methods, analysis confirms that the slope of the loop segments plays an important role in determining the size of the error.

We also ran experiments on the scheme that inserted hysteresis inversion in both the feedback and feedforward paths and compared its performance with other methods. PI-controller is used as the feedback controller in the proposed method. While it is known that the PI-controller (alone) has poor performance at high frequencies, our results show that when augmented with feedforward and when linear dynamics of the nanopositioner have large bandwidth, it performs better than other methods that we compared with.

By having a close look at how the hysteresis loop is shaped by different segments and considering changes of their slopes, a wide field is opened for future research. For instance, instead of using the conventional PI-controller, we can apply this method of analysis to more sophisticated robust and adaptive control approaches. Determining the upper-bound of the slope perturbation ($\Delta m$) would help
in designing these controllers. This bound would be less conservative than just considering the size of the complete hysteresis loop.

In future work we will conduct such analysis together with better approximation for the linear plant. In particular, we will more explicitly treat the case of non-zero $\epsilon$ for the singular perturbation analysis, to quantify the effect of the reference frequency on the tracking performance. Note that the analysis in this paper is preliminary because it looks at the error only over one segment. One of the tasks of future research is to extend the analysis to understand the cumulative effect of model uncertainty on the steady-state tracking error. This would be feasible and particularly relevant in the settings of periodic references, where the results could shed key insight into how the tracking error scales with the reference frequency.

**REFERENCES**


