Bounded Real Transfer Functions and G-Contractive Solutions of Matrix Riccati Differential Equations

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ABSTRACT

Bounded real systems represent an energy dissipating system. A mutual relationship between these systems and algebraic Riccati equations will be established. This relationship helps clarify the fundamental role that structural properties impart to the qualitative behavior of the algebraic Riccati equations. Strictly bounded real systems assure that the general solutions of the algebraic Riccati equation are contractive, which means that their eigenvalues are localized to the unit interval. In this paper a generalized contractive behavior is described for the solutions of Riccati differential equations, which defines a new equivalent condition between these equations and the dynamical systems.

INTRODUCTION

Probably one of the more important tasks in modern systems theory is to match the subtlety in the structure of multivariable systems with the qualitative behavior exhibited by the solutions of the corresponding Riccati equations. A major step in this endeavor was taken by Willems [1], who characterized Riccati solutions by utilizing matrix frequency domain inequalities and relations. Recently, Opdenacker and Jonckheere [2] exploited a direct corollary of Willems's work, to derive new balanced model reduction algorithms. His corollary established an equivalence between the "gap" behavior defined by the extremal solution of the algebraic Riccati equations and the contractive properties of bounded real stable transfer functions. In this note a new relation between bounded real stable transfer functions, all pass transfer functions and a generalized contractive property of solutions of Riccati differential equations is established. This relation provides another link between Riccati equations and the linear matrix inequality, further highlighting the basic importance of the linear matrix inequality as first noted by Willems [1]. Since bounded real problems can be mapped into positive real problems by the Cayley transform, the range of applications includes network analysis, optimal control and robustness analysis.

PRELIMINARIES

Consider the standard linear controllable/observable system

\[ \begin{align*}
\dot{x} &= Ax + Bu \\
y &= Cx + Du
\end{align*} \]  

(1)

where \( A \in \mathbb{R}^{n \times n} \), \( B \in \mathbb{R}^{n \times m} \), \( C \in \mathbb{R}^{k \times n} \) and \( D \in \mathbb{R}^{k \times m} \). The vectors \( x, u \) and \( y \) are compatibly dimensioned. The notation \( M \in \mathbb{R}^{m \times n} (M \in \mathbb{C}^{m \times n}) \) denotes an \( m \times n \) matrix with real (complex) coefficients. For any \( M \in \mathbb{C}^{n \times n} \) its spectrum (set of \( n \) eigenvalues) will be denoted by \( \lambda(M) \) and its spectral abscissa by \( \alpha(M) = \max \{ \Re: \lambda \in \Lambda(M) \} \). Let \( s \) be a complex variable and suppose that the transfer matrix \( Z(s) = D + C(sI - A)^{-1}B \) (see \( \Lambda(A) \)) satisfies the inequality

\[ Z^T(-jw)Z(jw) \leq 1. \]  

(3)

Here \( I \in \mathbb{R}^{n \times n} \) is the identity matrix, \( (sI - A)^{-1} \) denotes the matrix inverse, the superscript \( T \) denotes matrix transpose, \( j = \sqrt{-1} \) and \( w \in \mathbb{R} \). For any two matrices \( M, N \in \mathbb{R}^{n \times n} \), the matrix \( M \preceq N \) denotes that \( M - N \) is positive semi-definite, while \( J = \begin{pmatrix} 0 & I \\ -I & 0 \end{pmatrix} \) is the null matrix \( 0 \in \mathbb{R}^{2n \times 2n} \) with null matrix \( 0 \in \mathbb{R}^{2n \times 2n} \).

The inertia [3, p. 186] of a matrix \( M \in \mathbb{C}^{n \times n} \) denoted by \( \text{In}(M) \) is the triple \( (\pi(M), \nu(M), \delta(M)) \), where \( \pi(M) \), \( \nu(M) \), \( \delta(M) \) tally the number of eigenvalues with positive real parts, negative real parts and imaginary parts, respectively. Obviously, \( \pi(M) + \nu(M) + \delta(M) = n \) and the matrix \( M \) is nonsingular if \( \delta(M) = 0 \). For any two matrices \( M, N \in \mathbb{R}^{n \times n} \) the inequality \( M \geq N \) \( (M > N) \) denotes the usual partial order with \( \nu(M - N) = 0 \). The matrix \( M \) is congruent to \( N \) if there exists a nonsingular matrix \( T \) such that \( T^MT = N \). Congruent matrices have the same inertia.

A matrix \( E \in \mathbb{R}^{n \times n} \) is (strictly) intertwined with the symmetric nonsingular matrices \( G_1, G_2 \in \mathbb{R}^{n \times n} \) iff...
\[ e^{T}G_{1}E \leq G_{2} \quad (e^{T}G_{1}E < G_{2}) \]

and \( \ln(G_{1}) = \ln(G_{2}) \).

If \( G_{1} = G_{2} \), then \( E \) is a \( G_{1} \)-contraction. When \( E \) is strictly \( G_{1} \)-contractive, then the number of eigenvalues of \( E \) that are inside and outside the unit disk and are equal to \( \sigma_{1}(G_{1}) \) and \( \sigma_{1}(G_{1}) \), respectively. Moreover, if \( G_{1} \) is positive definite (i.e., \( \sigma_{1}(G_{1}) = \delta(G_{1}) = 0 \)), then all of the eigenvalues of \( E \) are inside the unit disk.

If strict inequality prevails in (3) then the transfer matrix \( Z(s) \) is strictly bounded real. It is bounded real iff it satisfies the linear matrix inequality

\[
\begin{bmatrix}
-(\dot{A}^{T}P + \dot{P}A + C^{T}C), & -\dot{P}B - C^{T}D \\
-B^{T}\dot{P} - D^{T}C, & R
\end{bmatrix} \succeq 0 \quad (>0) \text{ LMI}
\]

for some \( \dot{P} = \dot{P}^{T} > 0 \) with \( R = I - D^{T}D \). This block matrix is essentially equivalent to Willems's linear matrix inequality. The bounded real lemma [5] insures that (3) is equivalent to LMI for \( \alpha(A) < 0 \). When \( Z(s) \) is strictly bounded real and \( \alpha(A) < 0 \) their equivalence with strict inequality depends on a Riccati inequality established in Zhou and Khargonekar [6].

When \( R \) is nonsingular and \( \alpha(A) < 0 \), then the two extremal solutions \( P_{+} \) and \( P_{-} \) for the algebraic Riccati equation

\[
\Gamma_{C}(P) = A^{T}P + PA + C^{T}C + (PB + C^{T}D)R^{-1}(PB + C^{T}D)^{T},
\]

\[ \Gamma_{C}(P) = 0 \] (CARE)

are solutions of the LMI and any solution of the LMI satisfies \( 0 < P \leq P_{+} \). Note that \( P \) is the stabilizing solution of ARE [2], i.e., \( \alpha(A + BR^{-1}B^{T}P + D^{T}C)) < 0 \).

A dual LMI is obtained by the transformation \((A, B, C, D) \rightarrow (A^{T}, C^{T}, B^{T}, D^{T}) \) and the facts that if \( Z(s) \) is bounded real so is \( Z^{T}(s) \) and if \( \alpha(A) < 0 \) so is \( \alpha(A^{T}) < 0 \). The dual Riccati equation for CARE with \( S = I - DD^{T} \) is

\[
\Gamma_{C}(P) = AQ + QA^{T} + BB^{T} + (QC^{T} + BD^{T})S^{-1}(QC^{T} + BD^{T})^{T},
\]

\[ \Gamma_{C}(P) = 0 \] (FARE)

It too has two extremal solutions \( Q_{-} \) and \( Q_{+} \).\((0 \leq Q \leq Q_{+}) \) with stabilizing solution \( Q_{+} \) [2], i.e., \( \alpha(A + Q(C^{T} + BD^{T})S^{-1}C)) < 0 \). The \( nxn \) unique symmetric solutions \( Q(t, K) \) and \( P(t, K) \) with initial condition matrices \( K^{T} = K \) satisfy the differential Riccati equations

\[
\dot{P} = \Gamma_{C}(P), \quad P(0, K) = K \] (RDEC)

\[
\dot{Q} = \Gamma_{C}(Q), \quad Q(0, K) = K \] (RDEF)

Associated with RDEC and RDEF are the four block Hamiltonian matrices

\[
H_{C} = \begin{pmatrix}
-A^{T}, & -B^{T}C^{T} \\
C^{T}S^{-1}C, & A^{T}
\end{pmatrix}, \quad H_{P} = \begin{pmatrix}
-A^{T}, & -C^{T}S^{-1}C \\
B^{T}C^{T}, & A
\end{pmatrix}
\]

where \( A = A + BR^{-1}B^{T}D^{T} \). Recall, that a matrix \( H \) is Hamiltonian if it satisfies the equation \( H^{T}J + JH = 0 \).

Let \( \Phi(t, t_{0}) \) denote the state transition matrix of the linear differential equation

\[
\dot{z} = H_{C}z, \quad z(t_{0}) = z_{0} \quad (\Phi(t_{0}, t_{0}) = I @ 1)
\]

Unless otherwise noted \( t_{0} = 0 \) and the explicit dependency on \( t_{0} \) will be omitted. Partition \( \Phi(t) \) into four \( nxn \) blocks as

\[
\Phi(t) = \begin{pmatrix}
\Phi_{11}(t), & \Phi_{12}(t) \\
\Phi_{21}(t), & \Phi_{22}(t)
\end{pmatrix}
\]

It is well known that [7, p.156] as long as the inverse exists the solution of RDEC is

\[
P(t, K) = [\Phi_{21}(t) + \Phi_{22}(t)K][\Phi_{11}(t) + \Phi_{12}(t)K]^{-1}
\]

Since the pair for (1) \((A, B) \) is controllable, the pair \((\tilde{A}, BY_{C}) \) with \( I - D^{T}D = Y_{C}Y_{C}^{T} \) is controllable [8] and so the matrix \( \Phi_{11}(t) + \Phi_{12}(t)K \) is locally nonsingular on the interval \( 0 < t < \tilde{t} \) [9, p. 46]. Its singularity at \( \tilde{t} \) is equivalent to the RDEC having finite escape time. Similarly, since the pair for (2) \((C, A) \) is observable, the pair \((\tilde{A}^{T}, C^{T}Y_{C}^{T}) \) with \( I - D^{T}D = Y_{C}Y_{C}^{T} \) is controllable.

**RICCATI INTERTWINING SOLUTIONS**

By sharpening a result in Hewer and Kenney [10], the following intertwining theorem can be established. The corresponding theorems for the dual Riccati equation can be easily supplied by the interested reader.

**Theorem**

If \( G_{1} \) and \( G_{2} \) are symmetric nonsingular matrices such that the Lyapunov inequality

\[
H_{C}^{T}(-G_{1}G_{2} + (-G_{2}G_{1})H_{C} \leq 0
\]

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is satisfied and $\ln(G_1) = \ln(G_2)$, then for any initial condition matrix $K$ with $K^TG_2K < G_1$ the solution $P(t, K)$ is intertwined with $G_2$ and $G_1$ on some interval $0 < t < i(K)$ ($i(K) = \infty$ when $G_1 > 0$).

**Proof:**
For any vector $z \in \mathbb{R}^{2n}$ define the indefinite Lyapunov function

$$V(z) = z^T (-G_2 \otimes G_2) z.$$  \hfill (7)

Let $x(t, K, z_0)$ denote the $2n$ dimensional solution vector of (4) with component vectors $(\phi_{11}(t) + \phi_{12}(t)K)z_0$ and $(\phi_{21}(t) + \phi_{22}(t)K)z_0$. Now by standard Lyapunov theory it follows that the total derivative $V(z)$ satisfies the equation

$$\frac{dV(z)}{dt} = z^T [H^T\left(-G_2 \otimes G_2\right) + (-G_1 \otimes G_1) H_C] z$$

and so along the solution of (4)

$$V(x(t, K, z_0)) = \int_{z_0}^{x(t, K)} V(x(s, K, z_0)) ds.$$ \hfill (8)

As noted $(\phi_{11}(t) + \phi_{12}(t)K)$ is nonsingular on some interval $0 \leq t < i(K)$. Substituting the solution vector $x(t, K, z_0)$ into (7) and factoring the quadratic form with the aid of (5) leads to the following expression

$$V(z(t, K, z_0)) = x^T_0 [\phi_{11}(t) + \phi_{12}(t)K] \left[H(\phi_{11}(t) + \phi_{12}(t)K)\right] x_0$$

which establishes the first claim using (6) and (8).

Now suppose that $G_1 > 0$ and $K^TG_2K < G_1$, then by (8), the following inequality is valid on some interval $0 \leq t < i(K)$

$$0 \leq (\phi_{11}(t) + \phi_{12}(t)K)^T G_2 (\phi_{11}(t) + \phi_{12}(t)K) < (\phi_{11}(t) + \phi_{12}(t)K)^T G_2 (\phi_{11}(t) + \phi_{12}(t)K).$$

Since this inequality implies that $\phi_{11}(t) + \phi_{12}(t)K$ is nonsingular when $t = i(K)$, no finite escape time is possible on $0 \leq t < \infty$.

The two block matrices for the Lyapunov inequality (6) evaluated with respect to $H_C$ and $H_F$ are, respectively,

$$H_C(G_1, G_2) = \left[\begin{array}{c c c c}
\mathcal{A} G_1 + G_2 \mathcal{A} & C^{-T}CG_2 + G_2BR^{-T}B^T \\
G_2C^{-T}C + BR^{-T}B^T G_2 & \mathcal{A} G_2 + G_2 \mathcal{A}^T
\end{array}\right]$$

and

$$H_F(G_1, G_2) = \left[\begin{array}{c c c c}
\mathcal{A} G_2 + G_2 \mathcal{A}^T & BR^{-T}B^T G_1 + G_2C^{-T}C \\
G_2BR^{-T}B^T + C^T C & \mathcal{A} G_1 + G_1 \mathcal{A}
\end{array}\right].$$

Using the congruence transformation $(I@J)$ it is easy to show that $H_C(G_2, G_1)$ is congruent to $H_C(G_1, G_2)$, which establishes the following corollary.

**Corollary 1**

If $G_1$ and $G_2$ satisfy the conditions of Theorem 1, then for any initial condition matrix $K$ with $K^TG_2K < G_2$, the solution $Q(t, K)$ is intertwined with $G_1$ and $G_2$ on some interval $0 \leq t < i(K)$ and $(i(K) = \infty$ when $G_1 > 0$).

If $P(t, K)$ is intertwined with $G_2$ and $G_1$ and if $Q(t, K)$ is intertwined with $G_1$ and $G_2$, then the intertwining property is invariant under a nonsingular similarity transformation $(A, B, C, D)$.

The similarity invariance of the eigenvalues of the product of extremal solutions for the control and filtering algebraic Riccati equations was first established in Jonckheere and Silverman [11].

**BOUNDED REAL TRANSFER MATRICES AND G-CONTRACTION RICCATI SOLUTIONS**

The main result of this section is Theorem 3 which provides a new equivalent characterization of strictly bounded real transfer matrices with $a(A) < 0$.

For completeness two known important characterizations of these matrices are now summarized. The "gap" equivalence is found in Opdenacker and Jonckheere [2] and is a direct corollary of a result in Willems, [1, Theorem 5]. When $a(A) < 0$ the "gap" $P_+ - P_0 G_1 > 0$ of the CARE is strictly positive definite iff $Z(s)$ is strictly bounded real. The second equivalence does not require that $a(A) < 0$ and can be deduced from a theorem in Boyd et al. [12]. Their result is: $Z(s)$ is strictly bounded real iff $\delta(H_C) = \delta(H_F) = 0$ provided $\delta(A) = 0.$
For any \( M \in \mathbb{R}^{n \times n} \) the Moore-Penrose inverse \( M^+ \) is the unique solution of the matrix equations

\[
M^+ = M, \quad M^+ = M, \quad (MM)^T = MM, \quad (X MX)^T = XM
\]

**Theorem 2**

If \( R \) is nonsingular and \( Z(s) \) is (strictly) bounded real, then there exist intertwining matrices \( G_1 > 0 \) and \( G_2 > 0 \) such that the Lyapunov inequality (6)

\[
R_C^T(-G_1 \otimes G_2) + (-G_1 \otimes G_2) R_C \leq 0
\]

is (strictly) satisfied.

**Proof**

Let \( \hat{P} (\hat{P} > 0) \) be any solution of LMI and employ the congruence transformation

\[
\begin{bmatrix}
I & (PB + C^T D) R_C^{-1} T
\end{bmatrix} \in \mathbb{R}^{2n \times 2n}
\]

to show that the LMI inequality implies that \( \Gamma P \) is the unique solution of the matrix equations

\[
\begin{align*}
M_1 = M, & \quad M_2 = M, \\
(AX) = AX, & \quad (XB) = XB
\end{align*}
\]

Theorem 2 guarantees that the conditions of the Ostrowski-Schneider-Krein inertia theorem [3, p. 445] are satisfied, whence \( \Delta (H_C) = \Delta (H_F) \). Since \( \delta (G_1 \otimes G_2) = 0 \), the conclusion follows by Proposition 1.

These results lead to a new characterization of strictly bounded real systems.

**Theorem 3**

Suppose that \( \alpha (A) < 0 \). The transfer matrix \( Z(s) \) is strictly bounded real iff there exist positive definite matrices \( G_1 \) and \( G_2 \) such that \( P \) is strictly intertwined with \( G_1 \) and \( G_2 \), and \( Q \) is strictly intertwined with \( G_1 G_2 \).

**Proof**

Suppose that \( P, G_1, G_2 \) is strictly bounded real. Then the product \( Q, P \) is strictly contractive. Since the eigenvalues of the product are less than one and \( \alpha (S_i) \leq 0 \), it follows that \( P, P \geq 0 \) and so \( Z(s) \) is strictly bounded real by the "gap" test.

When \( Z(s) \) is strictly bounded real then Corollary 2 guarantees that \( \delta (H_C) = \Delta (H_F) \). Under these conditions Laub's construction [14] can be used to transform \( H_C \) into the four block matrix by an orthogonal matrix \( U \in \mathbb{R}^{n \times n} \) such that \( S_i = U_1 U_2 \) with \( S_i \geq 0 \) and \( \alpha (S_i) < 0 \). Conformally partition \( U \) into the four block matrix

\[
\begin{bmatrix}
U_{11} & U_{12} \\
U_{21} & U_{22}
\end{bmatrix}
\]

and note that \( \alpha (S_i) < 0 \). The equations

\[
\begin{align*}
\Gamma C U_2 U_1 & = \Gamma, \\
U_1 & = \Gamma U_1
\end{align*}
\]

imply by Theorem 6 in Molinari [15] that \( P \) is strictly contractive. Since the bracketed expression in the integrand is positive definite and the integral is convergent, it follows that \( P, G_1, G_2 \) is strictly bounded real by the "gap" test.

If \( R \) is nonsingular, then for any pair of matrices \( G_1 \) and \( G_2 \) \( \alpha (H_C(G_1, G_2)) = \Delta (H_F(G_1, G_2)) \).

**Proposition 1**

For any \( M \in \mathbb{R}^{n \times n} \) the Moore-Penrose inverse \( M^+ \) is the unique solution of the matrix equations

\[
M^+ = M, \quad M^+ = M, \quad (MM)^T = MM, \quad (X MX)^T = XM
\]

**Proof**

Using the congruence transformation \(-1 \otimes I\) guarantees that both Lyapunov matrices \( H_C(G_1, G_2) \) and \( H_F(G_2, G_1) \) share a common inertia property.

**Corollary 2**

If \( R \) is nonsingular and \( Z(s) \) is strictly bounded real, then \( \delta (H_C) = \delta (H_F) = 0 \).

**Proof**

The congruence transformation \(-1 \otimes I\) guarantees that both Lyapunov matrices \( H_C(G_1, G_2) \) and \( H_F(G_2, G_1) \) share a common inertia property.
Now these arguments can be repeated for $H_F$ by interchanging the roles of $G_2$ and $G_1$, recalling that $(\Lambda^T, C_T Y_F)$ is controllable and using Proposition 1.

The significance of Theorems 2 and 3 is threefold: first, they give a new characterization of bounded real functions and their relation to the Lyapunov inequality in equation (6), second, all other minimal realizations are of $Z(s)$ are intertwined, and finally, it shows that the growth rate of the general solutions $P(t, K)$ and $Q(t,K)$ are constrained by the intertwining inequalities and completely predictable from the bounded real property. While the solutions $P(t, 0)$ and $Q(t, 0)$ are monotone increasing and therefore constrained by the behavior of $P$. ($Q$), it is not immediate that both $P(t, K)$ ($Q(t,K)$) and $P$. ($Q$) are intertwined.

REFERENCES